

**PHY 101**

**MOTION IN ONE DIMENSION**

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# Motion in One Dimension

## 1. Position, Time and Displacement

The particle's location is specified by its coordinate, which will be denoted by  $x$  or  $y$ .

As the particle moves, its coordinate changes with the time,  $t$ . The change in position from  $x_1$  to  $x_2$  of the particle is the displacement  $\Delta x$ , with  $\Delta x = x_2 - x_1$ .

## 2. Average Velocity and Average Speed

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

The average speed of the particle is absolute value of the average velocity and is given by

$$\bar{s} = \frac{\text{Distance travelled}}{\Delta t}$$

When a particle has a displacement  $\Delta x$  in a change of time  $\Delta t$ , its average velocity *for that time interval* is

In general, the value of the average velocity for a moving particle depends on the initial and final times for which we have found the displacements.

### 2.1.3 Instantaneous Velocity and Speed

We can answer the question “how fast is a particle moving at a particular time  $t$ ?” by finding the instantaneous velocity. This is the limiting case of the average velocity when the time

interval  $\Delta t$  include the time  $t$  and is as small as we can imagine:

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad (2.3)$$

The instantaneous speed is the absolute value (magnitude) of the instantaneous velocity.

If we make a plot of  $x$  vs.  $t$  for a moving particle the instantaneous velocity is the slope of the tangent to the curve at any point.

### 2.1.4 Acceleration

When a particle's velocity changes, then we say that the particle undergoes an acceleration.

If a particle's velocity changes from  $v_1$  to  $v_2$  during the time interval  $t_1$  to  $t_2$  then we define the average acceleration as

$$a = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \quad (2.4)$$

As with velocity it is usually more important to think about the instantaneous acceleration, given by

$$a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \quad (2.5)$$

If the acceleration  $a$  is positive it means that the velocity is instantaneously *increasing*; if  $a$  is negative, then  $v$  is instantaneously *decreasing*. Oftentimes we will encounter the word deceleration in a problem. This word is used when the sense of the acceleration is opposite that of the instantaneous velocity (the motion). Then the *magnitude* of acceleration is given, with its direction being understood.

### 2.1.5 Constant Acceleration

A very useful *special case* of accelerated motion is the one where the acceleration  $a$  is constant. For this case, one can show that the following are true:

$$v = v_0 + at \quad (2.6)$$

$$x = x_0 + v_0t + \frac{1}{2}at^2 \quad (2.7)$$

$$v^2 = v_0^2 + 2a(x - x_0) \quad (2.8)$$

$$x = x_0 + \frac{1}{2}(v_0 + v)t \quad (2.9)$$

In these equations, we mean that the particle has position  $x_0$  and velocity  $v_0$  at time  $t = 0$ ; it has position  $x$  and velocity  $v$  at time  $t$ .

These equations are valid *only* for the case of constant acceleration.

### 2.1.6 Free Fall

An object tossed up or down near the surface of the earth has a constant downward acceleration of magnitude  $9.80 \frac{\text{m}}{\text{s}^2}$ . This number is always denoted by  $g$ . Be very careful about the sign; in a coordinate system where the  $y$  axis points straight up, the acceleration of a freely-falling object is

$$a_y = -9.80 \frac{\text{m}}{\text{s}^2} = -g \quad (2.10)$$

Here we are assuming that the air has no effect on the motion of the falling object. For an object which falls for a long distance this can be a bad assumption.

Remember that an object in free-fall has an acceleration equal to  $-9.80 \frac{\text{m}}{\text{s}^2}$  while it is moving up, while it is moving down, while it is at maximum height... always!

## 2.2 Worked Examples

### 2.2.1 Average Velocity and Average Speed

1. Boston Red Sox pitcher Roger Clemens could routinely throw a fastball at a horizontal speed of  $160 \frac{\text{km}}{\text{hr}}$ . How long did the ball take to reach home plate  $18.4\text{m}$  away?

We assume that the ball moves in a horizontal straight line with an average speed of  $160 \text{ km/hr}$ . Of course, in reality this is not quite true for a thrown baseball.

We are given the average velocity of the ball's motion and also a particular displacement, namely  $\Delta x = 18.4\text{m}$ . Equation 2.1 gives us:

$$\bar{v} = \frac{\Delta x}{\Delta t} \quad \Rightarrow \quad \Delta t = \frac{\Delta x}{\bar{v}}$$

But before using it, it might be convenient to change the units of  $\bar{v}$ . We have:

$$\bar{v} = 160 \frac{\text{km}}{\text{hr}} \cdot \frac{1000\text{m}}{1 \text{ km}} \cdot \frac{1 \text{ hr}}{3600\text{s}} = 44.4 \frac{\text{m}}{\text{s}}$$

Then we find:

$$\Delta t = \frac{\Delta x}{\bar{v}} = \frac{18.4\text{m}}{44.4 \frac{\text{m}}{\text{s}}} = 0.414\text{s}$$

The ball takes  $0.414$  seconds to reach home plate.

## 2.2.2 Acceleration

2. An electron moving along the  $x$  axis has a position given by  $x = (16te^{-t})$  m, where  $t$  is in seconds. How far is the electron from the origin when it momentarily stops?

To find the velocity of the electron as a function of time, take the first derivative of  $x(t)$ :

$$v = \frac{dx}{dt} = 16e^{-t} - 16te^{-t} = 16e^{-t}(1 - t) \frac{\text{m}}{\text{s}}$$

again where  $t$  is in seconds, so that the units for  $v$  are  $\frac{\text{m}}{\text{s}}$ .

Now the electron “momentarily stops” when the velocity  $v$  is zero. From our expression for  $v$  we see that this occurs at  $t = 1$  s. At this particular time we can find the value of  $x$ :

$$x(1 \text{ s}) = 16(1)e^{-1} \text{ m} = 5.89 \text{ m}$$

The electron was 5.89m from the origin when the velocity was zero.

3. (a) If the position of a particle is given by  $x = 20t - 5t^3$ , where  $x$  is in meters and  $t$  is in seconds, when if ever is the particle’s velocity zero? (b) When is its acceleration  $a$  zero? (c) When is  $a$  negative? Positive? (d) Graph  $x(t)$ ,  $v(t)$ , and  $a(t)$ .

(a) From Eq. 2.3 we find  $v(t)$  from  $x(t)$ :

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(20t - 5t^3) = 20 - 15t^2$$

where, if  $t$  is in seconds then  $v$  will be in  $\frac{\text{m}}{\text{s}}$ . The velocity  $v$  will be zero when

$$20 - 15t^2 = 0$$

which we can solve for  $t$ :

$$15t^2 = 20 \quad \Rightarrow \quad t^2 = \frac{20}{15} = 1.33 \text{ s}^2$$

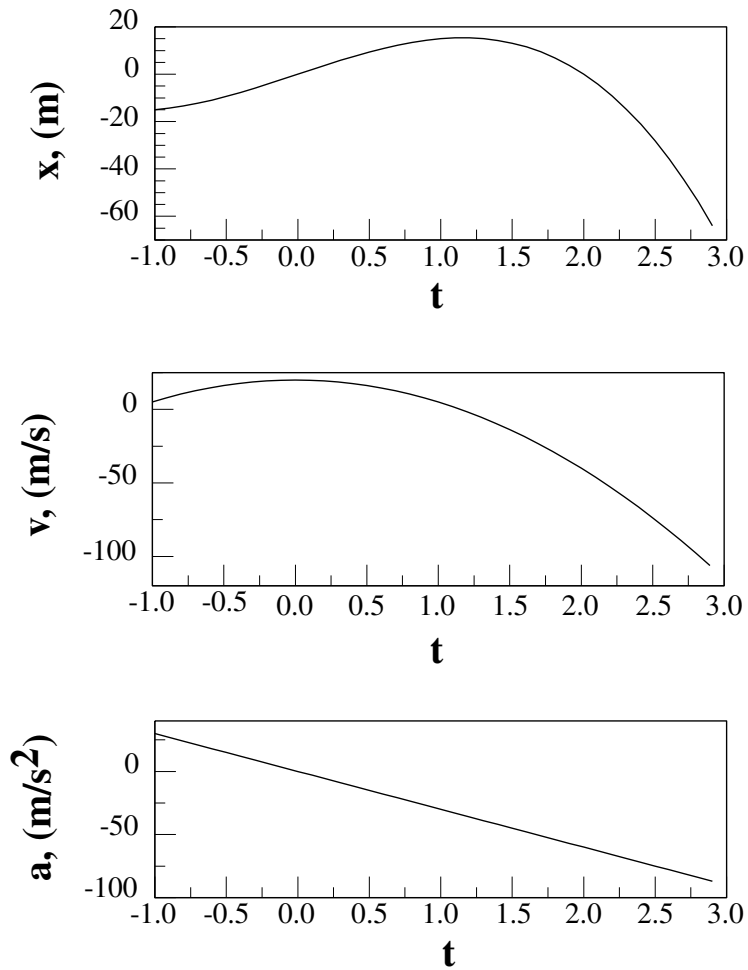
(The units  $\text{s}^2$  were inserted since we know  $t^2$  must have these units.) This gives:

$$t = \pm 1.15 \text{ s}$$

(We should be careful...  $t$  may be meaningful for negative values!)

(b) From Eq. 2.5 we find  $a(t)$  from  $v(t)$ :

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(20 - 15t^2) = -30t$$

Figure 2.1: Plot of  $x(t)$ ,  $v(t)$ , and  $a(t)$  for Example 3.

where we mean that if  $t$  is given in seconds,  $a$  is given in  $\frac{\text{m}}{\text{s}^2}$ . From this, we see that  $a$  can be zero only at  $t = 0$ .

(c) From the result in part (b) we can also see that  $a$  is negative whenever  $t$  is positive.  $a$  is positive whenever  $t$  is negative (again, assuming that  $t < 0$  has meaning for the motion of this particle).

(d) Plots of  $x(t)$ ,  $v(t)$ , and  $a(t)$  are given in Fig. 2.1.

4. In an arcade video game a spot is programmed to move across the screen according to  $x = 9.00t - 0.750t^3$ , where  $x$  is distance in centimeters measured from the left edge of the screen and  $t$  is time in seconds. When the spot reaches a screen edge, at either  $x = 0$  or  $x = 15.0\text{cm}$ ,  $t$  is reset to 0 and the spot starts moving again according to  $x(t)$ . (a) At what time after starting is the spot instantaneously at rest? (b) Where does this occur? (c) What is its acceleration when this occurs? (d) In what direction is it moving just prior to coming to rest? (e) Just after? (f) When does it first reach an edge of the screen after  $t = 0$ ?

(a) This is a question about the instantaneous velocity of the spot. To find  $v(t)$  we calculate:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(9.00t - 0.750t^3) = 9.00 - 2.25t^2$$

where this expression will give the value of  $v$  in  $\frac{\text{cm}}{\text{s}}$  when  $t$  is given in seconds.

We want to know the value of  $t$  for which  $v$  is zero, i.e. the spot is instantaneously at rest. We solve:

$$9.00 - 2.25t^2 = 0 \quad \Rightarrow \quad t^2 = \frac{9.00}{2.25} = 4.00\text{s}^2$$

(Here we have filled in the proper units for  $t^2$  since by laziness they were omitted from the first equations!) The solutions to this equation are

$$t = \pm 2.00\text{s}$$

but since we are only interested in times after the clock starts at  $t = 0$ , we choose  $t = 2.00\text{s}$ .

(b) In this part we are to find the value of  $x$  at which the instantaneous velocity is zero. In part (a) we found that this occurred at  $t = 2.00\text{s}$  so we calculate the value of  $x$  at  $t = 2.00\text{s}$ :

$$x(2.00\text{s}) = 9.00 \cdot (2.00) - 0.750 \cdot (2.00)^3 = 12.0\text{cm}$$

(where we have filled in the units for  $x$  since *centimeters* are implied by the equation). The dot is located at  $x = 12.0\text{cm}$  at this time. (And recall that the width of the screen is  $15.0\text{cm}$ .)

(c) To find the (instantaneous) acceleration at all times, we calculate:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(9.00 - 2.25t^2) = -4.50t$$

where we mean that if  $t$  is given in seconds,  $a$  will be given in  $\frac{\text{m}}{\text{s}^2}$ . At the time in question ( $t = 2.00\text{s}$ ) the acceleration is

$$a(t = 2.00\text{s}) = -4.50 \cdot (2.00) = -9.00$$

that is, the acceleration at this time is  $-9.00 \frac{\text{m}}{\text{s}^2}$ .

(d) From part (c) we note that at the time that the velocity was instantaneously zero the acceleration was *negative*. This means that the velocity was *decreasing* at the time. If the velocity was decreasing yet instantaneously equal to zero then it had to be going from positive to negative values at  $t = 2.00\text{s}$ . So just before this time its velocity was positive.

(e) Likewise, from our answer to part (d) just after  $t = 2.00\text{s}$  the velocity of particle had to be negative.

(f) We have seen that the dot never gets to the right edge of the screen at  $x = 15.0\text{cm}$ . It will not reverse its velocity again since  $t = 2.00\text{s}$  is the only positive time at which  $v = 0$ . So it will keep moving to back to the left, and the coordinate  $x$  will equal zero when we have:

$$x = 0 = 9.00t - 0.750t^3$$

Factor out  $t$  to solve:

$$t(9.00 - 0.750t^2) = 0 \quad \Rightarrow \quad \begin{array}{l} t = 0 \\ (9.00 - 0.750t^2) = 0 \end{array} \quad \begin{array}{l} \text{or} \\ \text{otherwise.} \end{array}$$

The first solution is the time that the dot started moving, so that is not the one we want. The second case gives:

$$(9.00 - 0.750t^2) = 0 \quad \Rightarrow \quad t^2 = \frac{9.00}{0.750} = 12.0\text{s}^2$$

which gives

$$t = 3.46\text{s}$$

since we only want the positive solution. So the dot returns to  $x = 0$  (the left side of the screen) at  $t = 3.46\text{s}$ .

If we plot the original function  $x(t)$  we get the curve given in Fig. 2.2 which shows that the spot does not get to  $x = 15.0\text{cm}$  before it turns around. (However as explained in the problem, the curve does *not* extend to negative values as the graph indicates.)

### 2.2.3 Constant Acceleration

5. The head of a rattlesnake can accelerate  $50 \frac{\text{m}}{\text{s}^2}$  in striking a victim. If a car could do as well, how long would it take to reach a speed of  $100 \frac{\text{km}}{\text{hr}}$  from rest?

First, convert the car's final speed to SI units to make it easier to work with:

$$100 \frac{\text{km}}{\text{hr}} = 100 \frac{\text{km}}{\text{hr}} \cdot \frac{1000\text{m}}{1\text{km}} \cdot \frac{1\text{hr}}{3600\text{s}} = 27.8 \frac{\text{m}}{\text{s}}$$



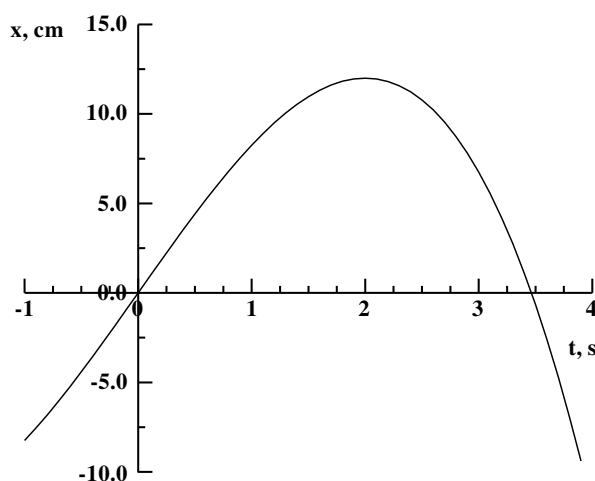


Figure 2.2: Plot of  $x$  vs  $t$  for moving spot. Ignore the parts where  $x$  is negative!

The acceleration of the car is  $50 \frac{\text{m}}{\text{s}^2}$  and it starts from rest which means that  $v_0 = 0$ . As we've found, the final velocity  $v$  of the car is  $27.8 \frac{\text{m}}{\text{s}}$ . (The problem actually that this is final *speed* but if our coordinate system points in the same direction as the car's motion, these are the same thing.) Equation 2.6 lets us solve for the time  $t$ :

$$v = v_0 + at \quad \Rightarrow \quad t = \frac{v - v_0}{a}$$

Substituting, we find

$$t = \frac{27.8 \frac{\text{m}}{\text{s}} - 0}{50 \frac{\text{m}}{\text{s}^2}} = 0.55 \text{ s}$$

If a car had such a large acceleration, it would take 0.55 s to attain the given speed.

6. A body moving with uniform acceleration has a velocity of  $12.0 \frac{\text{cm}}{\text{s}}$  when its  $x$  coordinate is  $3.00 \text{ cm}$ . If its  $x$  coordinate  $2.00 \text{ s}$  later is  $-5.00 \text{ cm}$ , what is the magnitude of its acceleration?

In this problem we are given the initial coordinate ( $x = 3.00 \text{ cm}$ ), the initial velocity ( $v_0 = 12.0 \frac{\text{cm}}{\text{s}}$ ), the final  $x$  coordinate ( $x = -5.00 \text{ cm}$ ) and the elapsed time ( $2.00 \text{ s}$ ). Using Eq. 2.7 (since we are told that the acceleration *is* constant) we can solve for  $a$ . We find:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \quad \Rightarrow \quad \frac{1}{2} a t^2 = x - x_0 - v_0 t$$

Substitute things:

$$\frac{1}{2} a t^2 = -5.00 \text{ cm} - 3.00 \text{ cm} - 12.0 \frac{\text{cm}}{\text{s}} (2.00 \text{ s}) = -32.0 \text{ cm}$$

Solve for  $a$ :

$$a = \frac{2(-32.0 \text{ cm})}{t^2} = \frac{2(-32.0 \text{ cm})}{(2.00 \text{ s})^2} = -16.0 \frac{\text{cm}}{\text{s}^2}$$

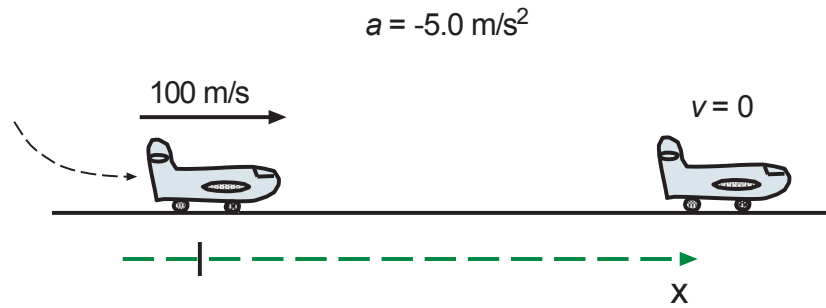


Figure 2.3: Plane touches down on runway at  $100 \frac{\text{m}}{\text{s}}$  and comes to a halt.

The  $x$  acceleration of the object is  $-16. \frac{\text{cm}}{\text{s}^2}$ . (The magnitude of the acceleration is  $16.0 \frac{\text{cm}}{\text{s}^2}$ .)

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7. A jet plane lands with a velocity of  $100 \frac{\text{m}}{\text{s}}$  and can accelerate at a maximum rate of  $-5.0 \frac{\text{m}}{\text{s}^2}$  as it comes to rest. (a) From the instant it touches the runway, what is the minimum time needed before it stops? (b) Can this plane land at a small airport where the runway is  $0.80\text{km}$  long?

(a) The data given in the problem is illustrated in Fig. 2.3. The minus sign in the acceleration indicates that the sense of the acceleration is opposite that of the motion, that is, the plane is decelerating.

The plane will stop as quickly as possible if the acceleration *does* have the value  $-5.0 \frac{\text{m}}{\text{s}^2}$ , so we use this value in finding the time  $t$  in which the velocity changes from  $v_0 = 100 \frac{\text{m}}{\text{s}}$  to  $v = 0$ . Eq. 2.6 tells us:

$$t = \frac{v - v_0}{a}$$

Substituting, we find:

$$t = \frac{(0 - 100 \frac{\text{m}}{\text{s}})}{(-5.0 \frac{\text{m}}{\text{s}^2})} = 20\text{s}$$

The plane needs 20s to come to a halt.

(b) The plane also travels the shortest distance in stopping if its acceleration is  $-5.0 \frac{\text{m}}{\text{s}^2}$ . With  $x_0 = 0$ , we can find the plane's final  $x$  coordinate using Eq. 2.9, using  $t = 20\text{s}$  which we got from part (a):

$$x = x_0 + \frac{1}{2}(v_0 + v)t = 0 + \frac{1}{2}(100 \frac{\text{m}}{\text{s}} + 0)(20\text{s}) = 1000\text{m} = 1.0\text{km}$$

The plane must have at least  $1.0\text{km}$  of runway in order to come to a halt safely.  $0.80\text{km}$  is *not* sufficient.

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8. A drag racer starts her car from rest and accelerates at  $10.0 \frac{\text{m}}{\text{s}^2}$  for the entire distance of  $400\text{m}$  ( $\frac{1}{4}$  mile). (a) How long did it take the car to travel this distance? (b) What is the speed at the end of the run?

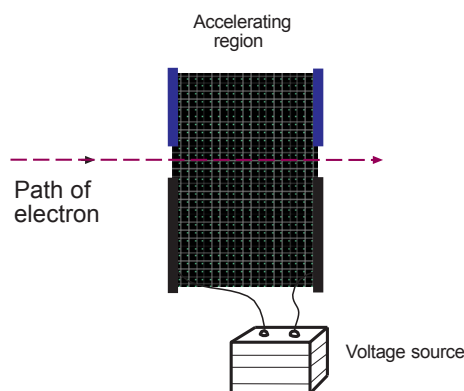


Figure 2.4: Electron is accelerated in a region between two plates, in Example 9.

(a) The racer moves in one dimension (along the  $x$  axis, say) with constant acceleration  $a = 10.0 \frac{\text{m}}{\text{s}^2}$ . We can take her initial coordinate to be  $x_0 = 0$ ; she starts from rest, so that  $v_0 = 0$ . Then the location of the car ( $x$ ) is given by:

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2} a t^2 \\ &= 0 + 0 + \frac{1}{2} a t^2 = \frac{1}{2} (10.0 \frac{\text{m}}{\text{s}^2}) t^2 \end{aligned}$$

We want to know the time at which  $x = 400\text{m}$ . Substitute and solve for  $t$ :

$$400\text{m} = \frac{1}{2} (10.0 \frac{\text{m}}{\text{s}^2}) t^2 \quad \Rightarrow \quad t^2 = \frac{2(400\text{m})}{(10.0 \frac{\text{m}}{\text{s}^2})} = 80.0\text{s}^2$$

which gives

$$t = 8.94\text{s} .$$

The car takes 8.94s to travel this distance.

(b) We would like to find the velocity at the end of the run, namely at  $t = 8.94\text{s}$  (the time we found in part (a)). The velocity is:

$$\begin{aligned} v &= v_0 + at \\ &= 0 + (10.0 \frac{\text{m}}{\text{s}^2}) t = (10.0 \frac{\text{m}}{\text{s}^2}) t \end{aligned}$$

At  $t = 8.94\text{s}$ , the velocity is

$$v = (10.0 \frac{\text{m}}{\text{s}^2})(8.94\text{s}) = 89.4 \frac{\text{m}}{\text{s}}$$

The speed at the end of the run is  $89.4 \frac{\text{m}}{\text{s}}$ .

9. An electron with initial velocity  $v_0 = 1.50 \times 10^5 \frac{\text{m}}{\text{s}}$  enters a region 1.0cm long where it is electrically accelerated, as shown in Fig. 2.4. It emerges with velocity  $v = 5.70 \times 10^6 \frac{\text{m}}{\text{s}}$ . What was its acceleration, assumed constant? (Such a process

occurs in the electron gun in a cathode-ray tube, used in television receivers and oscilloscopes.)

We are told that the acceleration of the electron is constant, so that Eqs. 2.6–2.9 can be used.

Here we know the initial and final velocities of the electron ( $v_0$  and  $v$ ). If we let its initial coordinate be  $x_0 = 0$  then the final coordinate is  $x = 1.0 \text{ cm} = 1.0 \times 10^{-2} \text{ m}$ . We don't know the time  $t$  for its travel through the accelerating region and of course we don't know the (constant) acceleration, which is what we're being asked in this problem.

We see that we can solve for  $a$  if we use Eq. 2.8:

$$v^2 = v_0^2 + 2a(x - x_0) \quad \Rightarrow \quad a = \frac{v^2 - v_0^2}{2(x - x_0)}$$

Substitute and get:

$$\begin{aligned} a &= \frac{(5.70 \times 10^6 \frac{\text{m}}{\text{s}})^2 - (1.50 \times 10^5 \frac{\text{m}}{\text{s}})^2}{2(1.0 \times 10^{-2} \text{ m})} \\ &= 1.62 \times 10^{15} \frac{\text{m}}{\text{s}^2} \end{aligned}$$

The acceleration of the electron is  $1.62 \times 10^{15} \frac{\text{m}}{\text{s}^2}$  (while it is in the accelerating region).

10. A world's land speed record was set by Colonel John P. Stapp when on March 19, 1954 he rode a rocket-propelled sled that moved down a track at  $1020 \frac{\text{km}}{\text{h}}$ . He and the sled were brought to a stop in 1.4s. What acceleration did he experience? Express your answer in  $g$  units.

For the period of deceleration of the rocket sled (which lasts for 1.4s) we are given the initial velocity and the final velocity, which is **zero** since the sled comes to rest at the end.

First, convert his initial velocity to SI units:

$$v_0 = 1020 \frac{\text{km}}{\text{h}} = (1020 \frac{\text{km}}{\text{h}}) \frac{10^3 \text{ m}}{1 \text{ km}} \frac{1 \text{ h}}{3600 \text{ s}} = 283.3 \frac{\text{m}}{\text{s}}$$

The Eq. 2.6 gives us the acceleration  $a$ :

$$v = v_0 + at \quad \Rightarrow \quad a = \frac{v - v_0}{t}$$

Substitute:

$$a = \frac{0 - 283.3 \frac{\text{m}}{\text{s}}}{1.4 \text{ s}} = -202.4 \frac{\text{m}}{\text{s}^2}$$

The acceleration is a *negative* number since it is opposite to the sense of the motion; it is a *deceleration*. The *magnitude* of the sled's acceleration is  $202.4 \frac{\text{m}}{\text{s}^2}$ .

To express this as a multiple of  $g$ , we note that

$$\frac{|a|}{g} = \frac{202.4 \frac{\text{m}}{\text{s}^2}}{9.80 \frac{\text{m}}{\text{s}^2}} = 20.7$$

so the magnitude of the acceleration was  $|a| = 20.7g$ . That's a lotta  $g$ 's!

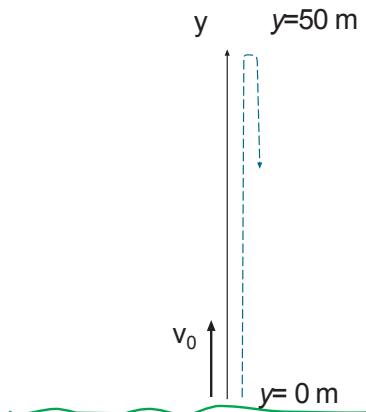


Figure 2.5: Object thrown upward reaches height of 50 m.

### 2.2.4 Free Fall

11. (a) With what speed must a ball be thrown vertically from ground level to rise to a maximum height of 50m? (b) How long will it be in the air?

(a) First, we decide on a coordinate system. I will use the one shown in Fig. 2.5, where the  $y$  axis points upward and the origin is at ground level. The ball starts its flight from ground level so its initial position is  $y_0 = 0$ . When the ball is at maximum height its coordinate is  $y = 50\text{m}$ , but *we also know its velocity at this point*. At maximum height the instantaneous velocity of the ball is **zero**. So if our “final” point is the time of maximum height, then  $v = 0$ .

So for the trip from ground level to maximum height, we know  $y_0$ ,  $y$ ,  $v$  and the acceleration  $a = -9.8 \frac{\text{m}}{\text{s}^2} = -g$ , but we *don't* know  $v_0$  or the time  $t$  to get to maximum height.

From our list of constant-acceleration equations, we see that Equation 2.8 will give us the initial velocity  $v_0$ :

$$v^2 = v_0^2 + 2a(y - y_0) \quad \Rightarrow \quad v_0^2 = v^2 - 2a(y - y_0)$$

Substitute, and get:

$$v_0^2 = (0)^2 - 2(-9.8 \frac{\text{m}}{\text{s}^2})(50\text{m} - 0) = 980 \frac{\text{m}^2}{\text{s}^2}$$

The next step is to “take the square root”. Since we know that  $v_0$  must be a *positive* number, we know that we should take the positive square root of  $980 \frac{\text{m}^2}{\text{s}^2}$ . We get:

$$v_0 = +31 \frac{\text{m}}{\text{s}}$$

The initial speed of the ball is  $31 \frac{\text{m}}{\text{s}}$ .

(b) We want to find the total time that the ball is in flight. What do we know about the ball when it returns to earth and hits the ground? We know that its  $y$  coordinate is equal to zero. (So far, we don't know anything about the ball's *velocity* at the time it returns to ground level.) If we consider the time between throwing and impact, then we do know  $y_0$ ,  $y$ ,  $v_0$  and of course  $a$ . If we substitute into Eq. 2.7 we find:

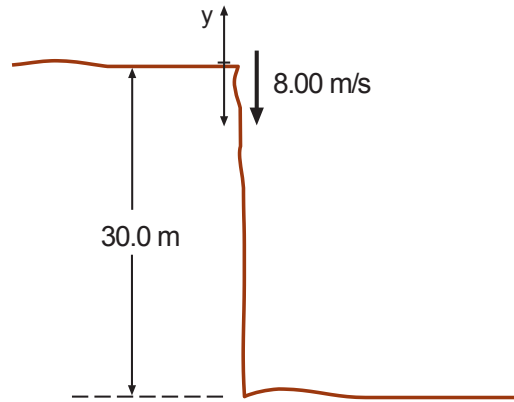


Figure 2.6: Ball is thrown straight down with speed of  $8.00 \frac{\text{m}}{\text{s}}$ , in Example 12.

$$0 = 0 + (31 \frac{\text{m}}{\text{s}})t + \frac{1}{2}(-9.8 \frac{\text{m}}{\text{s}^2})t^2$$

It is not hard to solve this equation for  $t$ . We can factor it to give:

$$t[(31 \frac{\text{m}}{\text{s}}) + \frac{1}{2}(-9.8 \frac{\text{m}}{\text{s}^2})t] = 0$$

which has two solutions. One of them is simply  $t = 0$ . This solution *is* an answer to the question we are asking, namely “When does  $y = 0$ ?” because the ball was at ground level at  $t = 0$ . But it is not the solution we want. For the other solution, we must have:

$$(31 \frac{\text{m}}{\text{s}}) + \frac{1}{2}(-9.8 \frac{\text{m}}{\text{s}^2})t = 0$$

which gives

$$t = \frac{2(31 \frac{\text{m}}{\text{s}})}{9.8 \frac{\text{m}}{\text{s}^2}} = 6.4\text{s}$$

The ball spends a total of 6.4 seconds in flight.

12. A ball is thrown directly downward with an initial speed of  $8.00 \frac{\text{m}}{\text{s}}$  from a height of 30.0m. When does the ball strike the ground?

We diagram the problem as in Fig. 2.6. We have to choose a coordinate system, and here I will put the let the origin of the  $y$  axis be at the place where the ball starts its motion (at the top of the 30 m height). With this choice, the ball starts its motion at  $y = 0$  and strikes the ground when  $y = -30\text{m}$ .

We can now see that the problem is asking us: At what time does  $y = -30.0\text{m}$ ? We have  $v_0 = -8.00 \frac{\text{m}}{\text{s}}$  (*minus* because the ball is thrown *downward!*) and the acceleration of the the ball is  $a = -g = -9.8 \frac{\text{m}}{\text{s}^2}$ , so at any time  $t$  the  $y$  coordinate is given by

$$y = y_0 + v_0t + \frac{1}{2}at^2 = (-8.00 \frac{\text{m}}{\text{s}})t - \frac{1}{2}gt^2$$

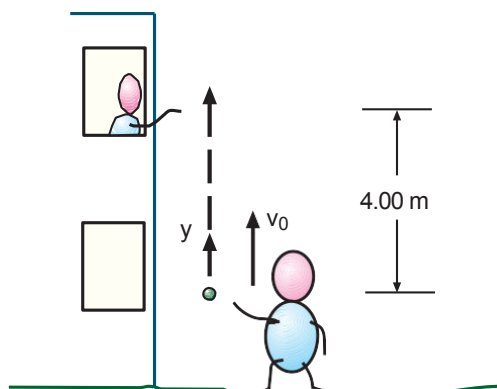


Figure 2.7: Student throws her keys into the air, in Example 13.

But at the time of impact we have

$$y = -30.0\text{m} = (-8.00 \frac{\text{m}}{\text{s}})t - \frac{1}{2}gt^2 = (-8.00 \frac{\text{m}}{\text{s}})t - (4.90 \frac{\text{m}}{\text{s}^2})t^2,$$

an equation for which we can solve for  $t$ . We rewrite it as:

$$(4.90 \frac{\text{m}}{\text{s}^2})t^2 + (8.00 \frac{\text{m}}{\text{s}})t - 30.0\text{m} = 0$$

which is just a quadratic equation in  $t$ . From our algebra courses we know how to solve this; the solutions are:

$$t = \frac{-(-8.00 \frac{\text{m}}{\text{s}}) \pm \sqrt{(-8.00 \frac{\text{m}}{\text{s}})^2 - 4(4.90 \frac{\text{m}}{\text{s}^2})(-30.0\text{m})}}{2(4.90 \frac{\text{m}}{\text{s}^2})}$$

and a little calculator work finally gives us:

$$t = \begin{array}{l} -3.42\text{s} \\ 1.78\text{s} \end{array}$$

Our answer is one of these ... which one? Obviously the ball had to strike the ground at some *positive* value of  $t$ , so the answer is  $t = 1.78\text{s}$ .

The ball strikes the ground 1.78s after being thrown.

13. A student throws a set of keys vertically upward to her sorority sister in a window 4.00 m above. The keys are caught 1.50 s later by the sister's outstretched hand. (a) With what initial velocity were the keys thrown? (b) What was the velocity of the keys just before they were caught?

(a) We draw a simple picture of the problem; such a simple picture is given in Fig. 2.7. Having a picture is important, but we should be careful not to put *too much* into the picture; the problem did not say that the keys were caught while they were going up or going down. For all we know at the moment, it could be either one!

We will put the origin of the  $y$  axis at the point where the keys were thrown. This simplifies things in that the initial  $y$  coordinate of the keys is  $y_0 = 0$ . Of course, since this is a problem about free-fall, we know the acceleration:  $\mathbf{a} = -\mathbf{g} = -9.80 \frac{\text{m}}{\text{s}^2}$ .

What mathematical information does the problem give us? We are told that when  $t = 1.50\text{s}$ , the  $y$  coordinate of the keys is  $y = 4.00\text{m}$ . Is this enough information to solve the problem? We write the equation for  $y(t)$ :

$$\begin{aligned} y &= y_0 + v_0 t + \frac{1}{2} a t^2 \\ &= v_0 t - \frac{1}{2} g t^2 \end{aligned}$$

where  $v_0$  is presently unknown. At  $t = 1.50\text{s}$ ,  $y = 4.00\text{m}$ , so:

$$4.00\text{m} = v_0(1.50\text{s}) - \frac{1}{2}(9.80 \frac{\text{m}}{\text{s}^2})(1.50\text{s})^2 .$$

Now we can solve for  $v_0$ . Rearrange this equation to get:

$$v_0(1.50\text{s}) = 4.00\text{m} + \frac{1}{2}(9.80 \frac{\text{m}}{\text{s}^2})(1.50\text{s})^2 = 15.0\text{m} .$$

So:

$$v_0 = \frac{15.0\text{m}}{1.50\text{s}} = 10.0 \frac{\text{m}}{\text{s}}$$

(b) We want to find the velocity of the keys at the time they were caught, that is, at  $t = 1.50\text{s}$ . We know  $v_0$ ; the velocity of the keys at all times follows from Eq. 2.6,

$$v = v_0 + at = 10.0 \frac{\text{m}}{\text{s}} - 9.80 \frac{\text{m}}{\text{s}^2} t$$

So at  $t = 1.50\text{s}$ ,

$$v = 10.0 \frac{\text{m}}{\text{s}} - 9.80 \frac{\text{m}}{\text{s}^2} (1.50\text{s}) = -4.68 \frac{\text{m}}{\text{s}} .$$

So the velocity of the keys when they were caught was  $-4.68 \frac{\text{m}}{\text{s}}$ . Note that the keys had a *negative* velocity; this tells us that the keys were moving *downward* at the time they were caught!

14. A ball is thrown vertically upward from the ground with an initial speed of  $15.0 \frac{\text{m}}{\text{s}}$ . (a) How long does it take the ball to reach its maximum altitude? (b) What is its maximum altitude? (c) Determine the velocity and acceleration of the ball at  $t = 2.00\text{s}$ .

(a) An illustration of the data given in this problem is given in Fig. 2.8. We measure the coordinate  $y$  upward from the place where the ball is thrown so that  $y_0 = 0$ . The ball's acceleration while in flight is  $\mathbf{a} = -\mathbf{g} = -9.80 \frac{\text{m}}{\text{s}^2}$ . We are given that  $v_0 = +15.0 \frac{\text{m}}{\text{s}}$ .

The ball is at maximum altitude when its (instantaneous) velocity  $v$  is *zero* (it is neither going up nor going down) and we can use the expression for  $v$  to solve for  $t$ :

$$v = v_0 + at \quad \Rightarrow \quad t = \frac{v - v_0}{a}$$



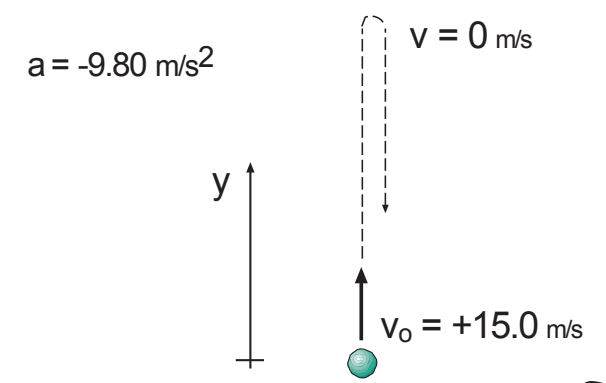


Figure 2.8: Ball is thrown straight up with initial speed  $15.0 \frac{\text{m}}{\text{s}}$ .

Plug in the values for the *top* of the ball's flight and get:

$$t = \frac{(0) - (15.0 \frac{\text{m}}{\text{s}})}{(-9.80 \frac{\text{m}}{\text{s}^2})} = 1.53 \text{ s} .$$

The ball takes 1.53s to reach maximum height.

(b) Now that we have the value of  $t$  when the ball is at maximum height we can plug it into Eq. 2.7 and find the value of  $y$  at this time and that will be the *value* of the maximum height. But we can also use Eq. 2.8 since we know all the values except for  $y$ . Solving for  $y$  we find:

$$v^2 = v_0^2 + 2ay \quad \Rightarrow \quad y = \frac{v^2 - v_0^2}{2a}$$

Plugging in the numbers, we get

$$y = \frac{(0)^2 - (15.0 \frac{\text{m}}{\text{s}})^2}{2(-9.80 \frac{\text{m}}{\text{s}^2})} = 11.5 \text{ m}$$

The ball reaches a maximum height of 11.5m.

(c) At  $t = 2.00\text{s}$  (that is, 2.0 seconds after the ball was thrown) we use Eq. 2.6 to find:

$$v = v_0 + at = (15.0 \frac{\text{m}}{\text{s}}) + (-9.80 \frac{\text{m}}{\text{s}^2})(2.00\text{s}) = -4.60 \frac{\text{m}}{\text{s}} .$$

so at  $t = 2.00\text{s}$  the ball is on its way back *down* with a speed of  $4.60 \frac{\text{m}}{\text{s}}$

As for the next part, the acceleration of the ball is *always* equal to  $-9.80 \frac{\text{m}}{\text{s}^2}$  while it is in flight.

15. A baseball is hit such that it travels straight upward after being struck by the bat. A fan observes that it requires 3.00s for the ball to reach its maximum height. Find (a) its initial velocity and (b) its maximum height. Ignore the effects of air resistance.

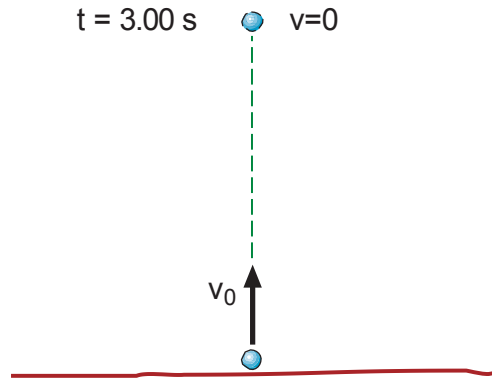


Figure 2.9: Ball is hit straight up; reaches maximum height 3.00s later.

(a) An illustration of the data given in the problem is given in Fig. 2.9.

For the period from when the ball is hit to the time it reaches maximum height, we know the time interval, the acceleration ( $a = -g$ ) and also the final velocity, since at maximum height the velocity of the ball is *zero*. Then Eq. 2.6 gives us  $v_0$ :

$$v = v_0 + at \quad \Rightarrow \quad v_0 = v - at$$

and we get:

$$v_0 = 0 - (-9.80 \frac{\text{m}}{\text{s}^2})(3.00\text{s}) = 29.4 \frac{\text{m}}{\text{s}}$$

The initial velocity of the ball was  $+29.4 \frac{\text{m}}{\text{s}}$ .

(b) To find the value of the maximum height, we need to find the value of the  $y$  coordinate at time  $t = 3.00\text{s}$ . We can use either Eq. 2.7 or Eq. 2.8. the latter gives:

$$v^2 = v_0^2 + 2a(y - y_0) \quad \Rightarrow \quad (y - y_0) = \frac{v^2 - v_0^2}{2a}$$

Plugging in the numbers we find that the change in  $y$  coordinate for the trip up was:

$$y - y_0 = \frac{0^2 - (29.4 \frac{\text{m}}{\text{s}})^2}{2(-9.80 \frac{\text{m}}{\text{s}^2})} = 44.1 \text{ m} .$$

The ball reached a maximum height of 44.1 m.

16. A parachutist bails out and freely falls 50m. Then the parachute opens, and thereafter she decelerates at  $2.0 \frac{\text{m}}{\text{s}^2}$ . She reaches the ground with a speed of  $3.0 \frac{\text{m}}{\text{s}}$ .

(a) How long was the parachutist in the air? (b) At what height did the fall begin?

(a) This problem gives several odd bits of information about the motion of the parachutist! We organize the information by *drawing a diagram*, like the one given in Fig. 2.10. It is very important to organize our work in this way!

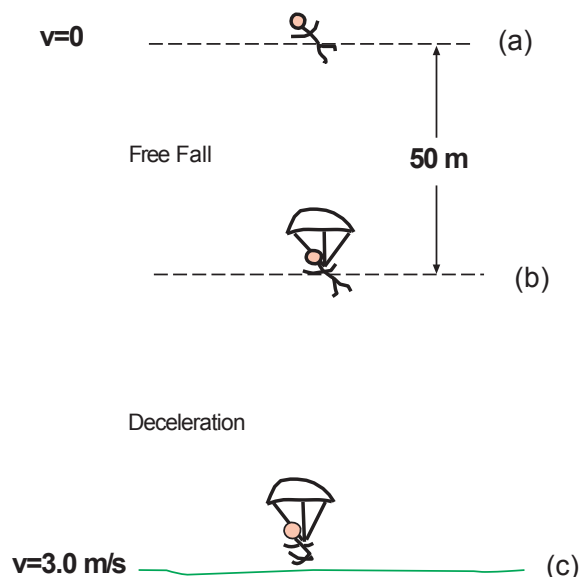


Figure 2.10: Diagram showing motion of parachutist in Example 16.

At the height indicated by (a) in the figure, the skydiver has *zero* initial speed. As she falls from (a) to (b) her acceleration is that of gravity, namely  $9.80 \frac{\text{m}}{\text{s}^2}$  *downward*. We know that (b) is 50m lower than (a) but we don't yet know the skydiver's speed at (b). Finally, at point (c) her speed is  $3.0 \frac{\text{m}}{\text{s}}$  and between (b) and (c) her "deceleration" was  $2.0 \frac{\text{m}}{\text{s}^2}$  but we don't know the difference in height between (b) and (c).

How can we start to fill in the gaps in our knowledge?

We note that on the trip from (a) to (b) we do know the starting velocity, the distance travelled and the acceleration. From Eq. 2.8 we can see that this is enough to find the final velocity, that is, the velocity at (b).

Use a coordinate system ( $y$ ) which has its origin at level (b), and the  $y$  axis pointing upward. Then the initial  $y$  coordinate is  $y_0 = 50\text{m}$  and the the initial velocity is  $v_0 = 0$ . The final  $y$  coordinate is  $y = 0$  and the acceleration is constant at  $a = -9.80 \frac{\text{m}}{\text{s}^2}$ . Then using Eq. 2.8 we have:

$$v^2 = v_0^2 + 2a(y - y_0) = 0 + 2(-9.80 \frac{\text{m}}{\text{s}^2})(0 - 50\text{m}) = 980 \frac{\text{m}^2}{\text{s}^2}$$

which has the solutions

$$v = \pm 31.3 \frac{\text{m}}{\text{s}}$$

but here the skydiver is obviously moving *downward* at (b), so we must pick

$$v = -31.3 \frac{\text{m}}{\text{s}}$$

for the velocity at (b).

While we're at it, we can find the time it took to get from (a) to (b) using Eq. 2.6, since we know the velocities and the acceleration for the motion. We find:

$$v = v_0 + at \quad \Rightarrow \quad t = \frac{v - v_0}{a}$$

Substitute:

$$t = \frac{(-31.3 \frac{\text{m}}{\text{s}} - 0)}{-9.80 \frac{\text{m}}{\text{s}^2}} = 3.19 \text{ s}$$

The skydiver takes 3.19 s to fall from (a) to (b).

Now we consider the motion from (b) to (c). For this part of the motion we know the initial and final velocities. We also know the acceleration, but we must be careful about how it is expressed. During this part of the trip, the skydiver's motion is always downward (velocity is always negative) but her speed decreases from  $31.9 \frac{\text{m}}{\text{s}}$  to  $3.0 \frac{\text{m}}{\text{s}}$ . The *velocity* changes from  $-31.3 \frac{\text{m}}{\text{s}}$  to  $-3.0 \frac{\text{m}}{\text{s}}$  so that the velocity has *increased*. The acceleration is *positive*; it is in the opposite sense as the motion and thus it was rightly called a "deceleration" in the problem. So for the motion from (b) to (c), we have

$$a = +2.0 \frac{\text{m}}{\text{s}^2}$$

We have the starting and final velocities for the trip from (b) to (c) so Eq. 2.6 lets us solve for the time  $t$ :

$$v = v_0 + at \quad \Rightarrow \quad t = \frac{v - v_0}{a}$$

Substitute:

$$t = \frac{-3.0 \frac{\text{m}}{\text{s}} - (-31.3 \frac{\text{m}}{\text{s}})}{+2.0 \frac{\text{m}}{\text{s}^2}} = 14.2 \text{ s}$$

Now we are prepared to answer part (a) of the problem. The time of the travel from (a) was 3.19 s; the time of travel from (b) to (c) was 14.2 s. The total time in the air was

$$t_{\text{Total}} = 3.19 \text{ s} + 14.2 \text{ s} = 17.4 \text{ s}$$

(b) Let's keep thinking about the trip from (b) to (c); we'll keep the origin at the same place as before (at (b)). Then for the trip from (b) to (c) the initial coordinate is  $y_0 = 0$ . The initial velocity is  $v_0 = -31.3 \frac{\text{m}}{\text{s}}$  and the final velocity is  $v = -3.0 \frac{\text{m}}{\text{s}}$ . We have the acceleration, so Eq. 2.8 gives us the final coordinate  $y$ :

$$v^2 = v_0^2 + 2a(y - y_0) \quad \Rightarrow \quad y - y_0 = \frac{v^2 - v_0^2}{2a}$$

Substitute:

$$y - y_0 = \frac{(-3.0 \frac{\text{m}}{\text{s}})^2 - (-31.3 \frac{\text{m}}{\text{s}})^2}{2(+2.0 \frac{\text{m}}{\text{s}^2})} = -243 \text{ m}$$

Since we chose  $y_0 = 0$ , the final coordinate of the skydiver is  $y = -243 \text{ m}$ .

We have used the same coordinate system in both parts, so overall the skydiver has gone from  $y = +50 \text{ m}$  to  $y = -243 \text{ m}$ . The change in height was

$$\Delta y = -243 \text{ m} - 50 \text{ m} = -293 \text{ m}$$

So the parachutist's fall began at a height of 293 m above the ground.

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17. A stone falls from rest from the top of a high cliff. a second stone is thrown downward from the same height 2.00s later with an initial speed of  $30.0 \frac{\text{m}}{\text{s}}$ . If both stones hit the ground simultaneously, how high is the cliff?

This is a “puzzle”-type problem which goes beyond the normal substitute-and-solve type; it involves more organization of our work and a clear understanding of our equations. Here’s the way I would attack it.

We have two different falling objects here with their own coordinates; we’ll put our origin at the top of the cliff and call the  $y$  coordinate of the first stone  $y_1$  and that of the second stone  $y_2$ . Each has a different dependence on the time  $t$ .

For the first rock, we have  $v_0 = 0$  since it falls from rest and of course  $a = -g$  so that its position is given by

$$y_1 = y_0 + v_0 t + \frac{1}{2} a t^2 = -\frac{1}{2} g t^2$$

This is simple enough but we need to remind ourselves that here  $t$  is the time since the *first* stone started its motion. It is *not* the same as the time since the *second* stone starts its motion. To be clear, let’s call this time  $t_1$ . So we have:

$$y_1 = -\frac{1}{2} g t_1^2 = -4.90 \frac{\text{m}}{\text{s}^2} t_1^2$$

Now, for the motion of the second stone, if we write  $t_2$  for the time since *it* started its motion, the facts stated in the problem tell us that its  $y$  coordinate is given by:

$$y_2 = y_0 + v_0 t_2 + \frac{1}{2} a t_2^2 = (-30.0 \frac{\text{m}}{\text{s}}) t_2 - \frac{1}{2} g t_2^2$$

So far, so good. The problem tells us that the first stone has been falling for 2.0s longer than the second one. This means that  $t_1$  is 2.0s larger than  $t_2$ . So:

$$t_1 = t_2 + 2.0\text{s} \quad \Rightarrow \quad t_2 = t_1 - 2.0\text{s}$$

(We will use  $t_1$  as our one time variable.) Putting this into our last equation and doing some algebra gives

$$\begin{aligned} y_2 &= (-30.0 \frac{\text{m}}{\text{s}})(t_1 - 2.0\text{s}) - \frac{1}{2}(9.80 \frac{\text{m}}{\text{s}^2})(t_1 - 2.0\text{s})^2 \\ &= (-30.0 \frac{\text{m}}{\text{s}})(t_1 - 2.0\text{s}) - (4.90 \frac{\text{m}}{\text{s}^2})(t_1^2 - 4.0\text{s}t_1 + 4.0\text{s}^2) \\ &= (-4.90 \frac{\text{m}}{\text{s}^2})t_1^2 + (-30.0 \frac{\text{m}}{\text{s}} + 19.6 \frac{\text{m}}{\text{s}})t_1 + (60.0\text{m} - 19.6\text{m}) \\ &= (-4.90 \frac{\text{m}}{\text{s}^2})t_1^2 + (-10.4 \frac{\text{m}}{\text{s}})t_1 + (40.4\text{m}) \end{aligned}$$

We need to remember that this expression for  $y_2$  will be meaningless for values of  $t_1$  which are less than 2.0 s. With this expression we can find values of  $y_1$  and  $y_2$  using the *same* time coordinate,  $t_1$ .

Now, the problem tells us that at some time ( $t_1$ ) the coordinates of the two stones are *equal*. We don’t yet know what that time or coordinate *is* but that is the information contained in the statement “both stones hit the ground simultaneously”. We can find this time by setting  $y_1$  equal to  $y_2$  and solving:

$$(-4.90 \frac{\text{m}}{\text{s}^2})t_1^2 = (-4.90 \frac{\text{m}}{\text{s}^2})t_1^2 + (-10.4 \frac{\text{m}}{\text{s}})t_1 + (40.4\text{m})$$

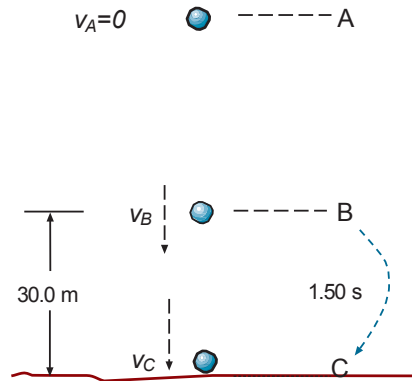


Figure 2.11: Diagram for the falling object in Example 18.

Fortunately the  $t^2$  term cancels in this equation making it a lot easier. We get:

$$(-10.4 \frac{\text{m}}{\text{s}})t_1 + (40.4\text{m}) = 0$$

which has the solution

$$t_1 = \frac{40.4\text{m}}{10.4 \frac{\text{m}}{\text{s}}} = 3.88\text{s}$$

So the rocks will have the same location at  $t_1 = 4.88\text{s}$ , that is, 4.88s after the first rock has been dropped.

What *is* that location? We can find this by using our value of  $t_1$  to get either  $y_1$  or  $y_2$  (the answer will be the same). Putting it into the expression for  $y_1$  we get:

$$y_1 = -4.90 \frac{\text{m}}{\text{s}^2} t^2 = (-4.90 \frac{\text{m}}{\text{s}^2})(4.88\text{s})^2 = -117\text{m}$$

So *both* stones were 117m below the initial point at the time of impact. The cliff is 117m high.

18. A falling object requires 1.50s to travel the last 30.0m before hitting the ground. From what height above the ground did it fall?

This is an intriguing sort of problem... very easy to state, but not so clear as to where to begin in setting it up!

The first thing to do is *draw a diagram*. We draw the important points of the object's motion, as in Fig. 2.11. The object has zero velocity at *A*; at *B* it is at a height of 30.0 m above the ground with an unknown velocity. At *C* it is at ground level, the time is 1.50 s later than at *B* and we also don't know the velocity here. Of course, we know the acceleration:  $a = -9.80 \frac{\text{m}}{\text{s}^2}$ !!

We are given all the information about the trip from *B* to *C*, so why not try to fill in our knowledge about this part? We know the final and initial coordinates, the acceleration and the time so we can find the initial velocity (that is, the velocity at *B*). Let's put the origin at ground level; then,  $y_0 = 30.0\text{m}$ ,  $y = 0$  and  $t = 1.50\text{s}$ , and using

$$y = y_0 + v_0 t + \frac{1}{2} a t^2$$

we find:

$$v_0 t = (y - y_0) - \frac{1}{2} a t^2 = (0 - (30.0\text{m})) - \frac{1}{2}(-9.80 \frac{\text{m}}{\text{s}^2})(1.50)^2 = -19.0\text{m}$$

so that

$$v_0 = \frac{(-19.0\text{m})}{t} = \frac{(-19.0\text{m})}{(1.50\text{s})} = -12.5 \frac{\text{m}}{\text{s}}.$$

This is the velocity at point *B*; we can also find the velocity at *C* easily, since that is the final velocity, *v*:

$$v = v_0 + a t = (-12.5 \frac{\text{m}}{\text{s}}) + (-9.80 \frac{\text{m}}{\text{s}^2})(1.50\text{s}) = -27.3 \frac{\text{m}}{\text{s}}$$

Now we can consider the trip from the starting point, *A* to the point of impact, *C*. We don't know the initial *y* coordinate, but we *do* know the final and initial velocities: The initial velocity is  $v_0 = 0$  and the final velocity is  $v = -27.3 \frac{\text{m}}{\text{s}}$ , as we just found. With the origin set at ground level, the final *y* coordinate is  $y = 0$ . We don't know the *time* for the trip, but if we use:

$$v^2 = v_0^2 + 2a(y - y_0)$$

we find:

$$(y - y_0) = \frac{(v^2 - v_0^2)}{2a} = \frac{(-27.3 \frac{\text{m}}{\text{s}})^2 - (0)^2}{2(-9.80 \frac{\text{m}}{\text{s}^2})} = -38.2\text{m}$$

and we can rearrange this to get:

$$y_0 = y + 38.2\text{m} = 0 + 38.2\text{m} = 38.2\text{m}$$

and so the object started falling from a height of 38.2 m.

There are probably cleverer ways to do this problem, but here I wanted to give you the slow, patient approach!

# Motion in Two and Three Dimensions

## 3.1 The Important Stuff

### 3.1.1 Position

In three dimensions, the location of a particle is specified by its location vector,  $\mathbf{r}$ :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (3.1)$$

If during a time interval  $\Delta t$  the position vector of the particle changes from  $\mathbf{r}_1$  to  $\mathbf{r}_2$ , the displacement  $\Delta \mathbf{r}$  for that time interval is

$$\Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad (3.2)$$

$$= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k} \quad (3.3)$$

### 3.1.2 Velocity

If a particle moves through a displacement  $\Delta \mathbf{r}$  in a time interval  $\Delta t$  then its average velocity for that interval is

$$\bar{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k} \quad (3.4)$$

As before, a more interesting quantity is the *instantaneous* velocity  $\mathbf{v}$ , which is the limit of the average velocity when we shrink the time interval  $\Delta t$  to zero. It is the time derivative of the position vector  $\mathbf{r}$ :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad (3.5)$$

$$= \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \quad (3.6)$$

$$= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \quad (3.7)$$

can be written:

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \quad (3.8)$$



where

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \quad (3.9)$$

The instantaneous velocity  $\mathbf{v}$  of a particle is always tangent to the path of the particle.

### 3.1.3 Acceleration

If a particle's velocity changes by  $\Delta \mathbf{v}$  in a time period  $\Delta t$ , the average acceleration  $\bar{\mathbf{a}}$  for that period is

$$\bar{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t} \mathbf{i} + \frac{\Delta v_y}{\Delta t} \mathbf{j} + \frac{\Delta v_z}{\Delta t} \mathbf{k} \quad (3.10)$$

but a much more interesting quantity is the result of shrinking the period  $\Delta t$  to zero, which gives us the instantaneous acceleration,  $\mathbf{a}$ . It is the time derivative of the velocity vector  $\mathbf{v}$ :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \quad (3.11)$$

$$= \frac{d}{dt}(v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}) \quad (3.12)$$

$$= \frac{dv_x}{dt} \mathbf{i} + \frac{dv_y}{dt} \mathbf{j} + \frac{dv_z}{dt} \mathbf{k} \quad (3.13)$$

which can be written:

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \quad (3.14)$$

where

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \quad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \quad (3.15)$$

### 3.1.4 Constant Acceleration in Two Dimensions

When the acceleration  $\mathbf{a}$  (for motion in two dimensions) is constant we have two sets of equations to describe the  $x$  and  $y$  coordinates, each of which is similar to the equations in Chapter 2. (Eqs. 2.6–2.9.) In the following, motion of the particle begins at  $t = 0$ ; the initial position of the particle is given by

$$\mathbf{r}_0 = x_0 \mathbf{i} + y_0 \mathbf{j}$$

and its initial velocity is given by

$$\mathbf{v}_0 = v_{0x} \mathbf{i} + v_{0y} \mathbf{j}$$

and the vector  $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$  is *constant*.

$$v_x = v_{0x} + a_x t \quad v_y = v_{0y} + a_y t \quad (3.16)$$

$$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \quad y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \quad (3.17)$$

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \quad v_y^2 = v_{0y}^2 + 2a_y(y - y_0) \quad (3.18)$$

$$x = x_0 + \frac{1}{2}(v_{0x} + v_x)t \quad y = y_0 + \frac{1}{2}(v_{0y} + v_y)t \quad (3.19)$$

Though the equations in each pair have the same *form* they are not identical because the components of  $\mathbf{r}_0$ ,  $\mathbf{v}_0$  and  $\mathbf{a}$  are not the same.

### 3.1.5 Projectile Motion

When a particle moves in a vertical plane during free-fall its acceleration is constant; the acceleration has magnitude  $9.80 \frac{\text{m}}{\text{s}^2}$  and is directed downward. If its coordinates are given by a horizontal  $x$  axis and a vertical  $y$  axis which is directed upward, then the acceleration of the projectile is

$$a_x = 0 \quad a_y = -9.80 \frac{\text{m}}{\text{s}^2} = -g \quad (3.20)$$

*For a projectile, the horizontal acceleration  $a_x$  is zero!!!*

Projectile motion is a special case of constant acceleration, so we simply use Eqs. 3.16–3.19, with the proper values of  $a_x$  and  $a_y$ .

### 3.1.6 Uniform Circular Motion

When a particle is moving in a circular path (or part of one) at *constant speed* we say that the particle is in uniform circular motion. Even though the speed is not changing, *the particle is accelerating* because its velocity  $v$  is changing *direction*.

The acceleration of the particle is directed toward the center of the circle and has magnitude

$$a = \frac{v^2}{r} \quad (3.21)$$

where  $r$  is the radius of the circular path and  $v$  is the (constant) speed of the particle. Because of the direction of the acceleration (i.e. toward the center), we say that a particle in uniform circular motion has a centripetal acceleration.

If the particle repeatedly makes a complete circular path, then it is useful to talk about the time  $T$  that it takes for the particle to make one complete trip around the circle. This is called the period of the motion. The period is related to the speed of the particle and radius of the circle by:

$$T = \frac{2\pi r}{v} \quad (3.22)$$

### 3.1.7 Relative Motion

The velocity of a particle depends on who is doing the measuring; as we see later on it is perfectly valid to consider “moving” observers who carry their own clocks and coordinate systems with them, i.e. they make measurements according to their own reference frame; that is to say, a set of Cartesian coordinates which may be in motion with respect to another set of coordinates. Here we will assume that the axes in the different system remain parallel to one another; that is, one system can move (translate) but not *rotate* with respect to another one.

Suppose observers in frames  $A$  and  $B$  measure the position of a point  $P$ . Then then if we have the definitions:

$r_{PA}$  = position of  $P$  as measured by  $A$

$r_{PB}$  = position of  $P$  as measured by  $B$

$r_{BA}$  = position of  $B$ 's origin, as measured by  $A$

with  $v$ 's and  $a$ 's standing for the appropriate time derivatives, then we have the relations:

$$r_{PA} = r_{PB} + r_{BA} \quad (3.23)$$

$$v_{PA} = v_{PB} + v_{BA} \quad (3.24)$$

For the purposes of doing physics, it is important to consider reference frames which move at *constant velocity* with respect to one another; for these cases,  $v_{BA} = \mathbf{0}$  and then we find that point  $P$  has the same acceleration in these reference frames:

$$a_{PA} = a_{PB}$$

Newton's Laws (next chapter!) apply to such a set of inertial reference frames. Observers in each of these frames agree on the value of a particle's acceleration.

Though the above rules for translation between reference frames seem very reasonable, it was the great achievement of Einstein with his theory of Special Relativity to understand the more subtle ways that we must relate measured quantities between reference frames. The trouble comes about because time ( $t$ ) is *not* the same absolute quantity among the different frames.

Among other places, Eq. 3.24 is used in problems where an object like a plane or boat has a known velocity in the frame of (with respect to) a medium like air or water which *itself* is moving with respect to the stationary ground; we can then find the velocity of the plane or boat with respect to the *ground* from the *vector sum* in Eq. 3.24.

## 3.2 Worked Examples

### 3.2.1 Velocity

1. The position of an electron is given by  $\mathbf{r} = 3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0t\mathbf{k}$  (where  $t$  is in seconds and the coefficients have the proper units for  $\mathbf{r}$  to be in meters). (a) What is  $\mathbf{v}(t)$  for the electron? (b) In unit-vector notation, what is  $\mathbf{v}$  at  $t = 2.0\text{s}$ ? (c) What are the magnitude and direction of  $\mathbf{v}$  just then?

(a) The velocity vector  $\mathbf{v}$  is the time-derivative of the position vector  $\mathbf{r}$ :

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0t\mathbf{k}) \\ &= 3.0\mathbf{i} - 8.0t\mathbf{j} \end{aligned}$$

where we mean that when  $t$  is in seconds,  $\mathbf{v}$  is given in  $\frac{\text{m}}{\text{s}}$ .

(b) At  $t = 2.00\text{s}$ , the value of  $v$  is

$$v(t = 2.00\text{s}) = 3.0\mathbf{i} - (8.0)(2.0)\mathbf{j} = 3.0\mathbf{i} - 16\mathbf{j}$$

that is, the velocity is  $(3.0\mathbf{i} - 16\mathbf{j}) \frac{\text{m}}{\text{s}}$ .

(c) Using our answer from (b), at  $t = 2.00\text{s}$  the magnitude of  $v$  is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{(3.00 \frac{\text{m}}{\text{s}})^2 + (-16. \frac{\text{m}}{\text{s}})^2 + (0)^2} = 16. \frac{\text{m}}{\text{s}}$$

we note that the velocity vector lies in the  $xy$  plane (even though this is a three-dimensional problem!) so that we can express its direction with a single angle, the usual angle  $\theta$  measured anti-clockwise in the  $xy$  plane from the  $x$  axis. For this angle we get:

$$\tan \theta = \frac{v_y}{v_x} = -5.33 \quad \Rightarrow \quad \theta = \tan^{-1}(-5.33) = -79^\circ .$$

When we take the inverse tangent, we should always check and see if we have chosen the right quadrant for  $\theta$ . In this case  $-79^\circ$  is correct since  $v_y$  is negative and  $v_x$  is positive.

### 3.2.2 Acceleration

2. A particle moves so that its position as a function of time in SI units is  $\mathbf{r} = \mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}$ . Write expressions for (a) its velocity and (b) its acceleration as functions of time.

(a) To clarify matters, what we mean here is that when we use the numerical value of  $t$  in *seconds*, we will get the values of  $\mathbf{r}$  in *meters*. Since the velocity vector is the time-derivative of the position vector  $\mathbf{r}$ , we have:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt}(\mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}) \\ &= 0\mathbf{i} + 8t\mathbf{j} + \mathbf{k} \end{aligned}$$

That is,  $\mathbf{v} = 8t\mathbf{j} + \mathbf{k}$ . Here, we mean that when we use the numerical value of  $t$  in seconds, we will get the value of  $\mathbf{v}$  in  $\frac{\text{m}}{\text{s}}$ .

(b) The acceleration  $\mathbf{a}$  is the time-derivative of  $\mathbf{v}$ , so using our result from part (a) we have:

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(8t\mathbf{j} + \mathbf{k}) \\ &= 8\mathbf{j} \end{aligned}$$

So  $a = 8j$ , where we mean that the value of  $a$  is in units of  $\frac{m}{s^2}$ . In fact, we should really include the units *here* and write:

$$a = 8 \frac{m}{s^2} j$$

3. A particle moving with an initial velocity  $v = (50 \frac{m}{s})j$  undergoes an acceleration  $a = [35m/s^2 + (2m/s^5)t^3]i + [4m/s^2 - (1m/s^4)t^2]j$ . What are the particle's position and velocity after 3.0s, assuming that it starts at the origin?

In the problem we are given the acceleration at *all* times, the *initial* velocity and also the *initial* position. We know that at  $t = 0$ , the velocity components are  $v_x = 0$  and  $v_y = 50 \frac{m}{s}$  and the coordinates are  $x = 0$  and  $y = 0$ .

From the acceleration  $a$  we do know *something* about the velocity. Since the acceleration is the time derivative of the velocity:

$$a = \frac{dv}{dt},$$

the velocity is the *anti-derivative* (or “indefinite integral”, “primitive”...) of the acceleration. Having learned our calculus well, we immediately write:

$$v = \frac{t^4}{35} + \frac{1}{2}t^4 + C_1 i + 4t - \frac{1}{3}t^3 + C_2 j$$

Here, for simplicity, I have omitted the units that are supposed to go with the coefficients. (I'm not supposed to do that!) Just keep in mind that time is supposed to be in *seconds*, length is in *meters*...

Of course, when we do the integration, we get constants  $C_1$  and  $C_2$  which (so far) have not been determined. We can determine them using the rest of the information in the problem. Since  $v_x = 0$  at  $t = 0$  we get:

$$35(0) + \frac{1}{2}(0)^4 + C_1 = 0 \quad \Rightarrow \quad C_1 = 0$$

and

$$4(0) - \frac{1}{3}(0)^3 + C_2 = 50 \quad \Rightarrow \quad C_2 = 50$$

so the velocity as a function of time is

$$v = \frac{t^4}{35} + \frac{1}{2}t^4 i + 4t - \frac{1}{3}t^3 + 50 j$$

where  $t$  is in seconds and the result is in  $\frac{m}{s}$ .

We can find  $r$  as a function of time in the same way. Since

$$v = \frac{dr}{dt}$$

then  $r$  is the anti-derivative of  $v$ . We get:

$$r = \frac{35}{2}t^2 + \frac{1}{10}t^5 + C_3 \mathbf{i} + 2t^2 - \frac{1}{12}t^4 + 50t + C_4 \mathbf{j}$$

and once again we need to solve for the constants.  $x = 0$  at  $t = 0$ , so

$$\frac{35}{2}(0)^2 + \frac{1}{10}(0)^5 + C_3 = 0 \quad \Rightarrow \quad C_3 = 0$$

and  $y = 0$  at  $t = 0$ , so

$$2(0)^2 - \frac{1}{12}(0)^4 + 50(0) + C_4 = 0 \quad \Rightarrow \quad C_4 = 0$$

and so  $r$  is fully determined:

$$r = \frac{35}{2}t^2 + \frac{1}{10}t^5 \mathbf{i} + 2t^2 - \frac{1}{12}t^4 + 50t \mathbf{j}$$

Now we can answer the questions.

We want to know the value of  $r$  (the particle's position) at  $t = 3.0\text{s}$ . Just plug in!

$$x(t = 3.0\text{s}) = \frac{35}{2}(3.0)^2 + \frac{1}{10}(3.0)^5 = 181\text{ m}$$

and

$$y(t = 3.0\text{s}) = 2(3.0)^2 - \frac{1}{12}(3.0)^4 + 50(3.0) = 161\text{ m}.$$

The components of the velocity at  $t = 3.0\text{s}$  are

$$v_x(t = 3.0\text{s}) = 35(3.0) + \frac{1}{2}(3.0)^4 = 146 \frac{\text{m}}{\text{s}}$$

and

$$v_y(t = 3.0\text{s}) = 4(3.0) - \frac{1}{3}(3.0)^3 + 50 = 53 \frac{\text{m}}{\text{s}}.$$

Here we have been careful to include the proper (SI) units in the final answers because coordinates and velocities must have *units*.

### 3.2.3 Constant Acceleration in Two Dimensions

4. A fish swimming in a horizontal plane has a velocity  $v_0 = (4\mathbf{i} + 1\mathbf{j}) \frac{\text{m}}{\text{s}}$  at a point in the ocean whose position vector is  $r_0 = (10\mathbf{i} - 4\mathbf{j})\text{m}$  relative to a stationary rock at the shore. After the fish swims with constant acceleration for  $20.0\text{s}$ , its velocity is  $v = (20\mathbf{i} - 5\mathbf{j}) \frac{\text{m}}{\text{s}}$ . (a) What are the components of the acceleration? (b) What is the direction of the acceleration with respect to the fixed  $x$  axis? (c) Where is the fish at  $t = 25\text{s}$  and in what direction is it moving?

(a) Since we are given that the acceleration is *constant*, we can use Eqs. 3.16:

$$v_x = v_{0x} + a_x t \quad v_y = v_{0y} + a_y t$$

to get:

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{(20.0 \frac{\text{m}}{\text{s}} - 4.0 \frac{\text{m}}{\text{s}})}{20.0 \text{s}} = 0.80 \frac{\text{m}}{\text{s}^2}$$

and

$$a_y = \frac{v_y - v_{0y}}{t} = \frac{(-5.0 \frac{\text{m}}{\text{s}} - 1.0 \frac{\text{m}}{\text{s}})}{20.0 \text{s}} = -0.30 \frac{\text{m}}{\text{s}^2}$$

and the acceleration *vector* of the fish is

$$\mathbf{a} = (0.80 \frac{\text{m}}{\text{s}^2})\mathbf{i} - (0.30 \frac{\text{m}}{\text{s}^2})\mathbf{j} .$$

(b) With the angle  $\theta$  measured counterclockwise from the  $+x$  axis, the direction of the acceleration  $\mathbf{a}$  is:

$$\tan \theta = \frac{a_y}{a_x} = \frac{-0.30}{0.80} = -0.375$$

A calculator gives us:

$$\theta = \tan^{-1}(-0.375) = -20.6^\circ$$

Since the  $y$  component of the acceleration is negative, this angle *is* in the proper quadrant. The direction of the acceleration is given by  $\theta = -20.6^\circ$ . (The same as  $\theta = 360^\circ - 20.6^\circ = 339.4^\circ$ .)

(c) We can use Eq. 3.17 to find the values of  $x$  and  $y$  at  $t = 25 \text{ s}$ :

$$\begin{aligned} x &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ &= 10\text{m} + 4.0 \frac{\text{m}}{\text{s}}(25 \text{s}) + \frac{1}{2}(0.80 \frac{\text{m}}{\text{s}^2})(25 \text{s})^2 \\ &= 360\text{m} \end{aligned}$$

and

$$\begin{aligned} y &= y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ &= -4.0\text{m} + 1.0 \frac{\text{m}}{\text{s}}(25 \text{s}) + \frac{1}{2}(-0.30 \frac{\text{m}}{\text{s}^2})(25 \text{s})^2 \\ &= -72.8\text{m} \end{aligned}$$

At  $t = 25 \text{ s}$  the velocity components of the fish are given by:

$$\begin{aligned} v_x &= v_{0x} + a_x t \\ &= 4.0 \frac{\text{m}}{\text{s}} + (0.80 \frac{\text{m}}{\text{s}^2})(25 \text{s}) = 24 \frac{\text{m}}{\text{s}} \end{aligned}$$

and

$$\begin{aligned} v_y &= v_{0y} + a_y t \\ &= 1.0 \frac{\text{m}}{\text{s}} + (-0.30 \frac{\text{m}}{\text{s}^2})(25 \text{s}) = -6.5 \frac{\text{m}}{\text{s}} \end{aligned}$$

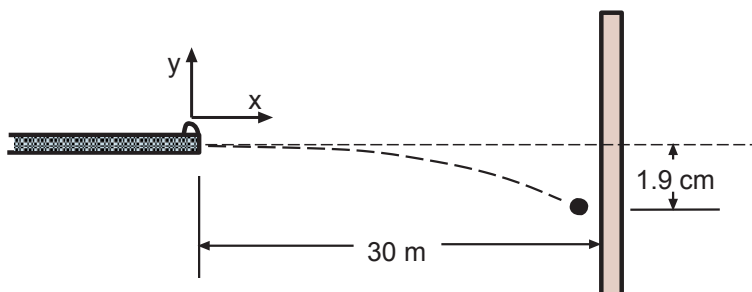


Figure 3.1: Bullet hits target 1.9 cm below the aiming point.

so that at that time the speed of the fish is

$$v = \sqrt{v_x^2 + v_y^2}$$

$$v^2 = (24 \frac{\text{m}}{\text{s}})^2 + (-6.5 \frac{\text{m}}{\text{s}})^2 = 24.9 \frac{\text{m}^2}{\text{s}^2}$$

and the direction of its motion  $\theta$  is found from:

$$\tan \theta = \frac{v_y}{v_x} = \frac{-6.5}{24} = -0.271$$

so that

$$\theta = -15.2^\circ$$

Again, since  $v_y$  is negative and  $v_x$  is positive, this is the correct choice for  $\theta$ . So the direction of the fish's motion is  $-15.2^\circ$  from the  $+x$  axis.

### 3.2.4 Projectile Motion

5. A rifle is aimed horizontally at a target 30 m away. The bullet hits the target 1.9 cm below the aiming point. (a) What is the bullet's time of flight? (b) What is the muzzle velocity?

(a) First, we define our coordinates. I will use the coordinate system indicated in Fig. 3.1, where the origin is placed at the tip of the gun. Then we have  $x_0 = 0$  and  $y_0 = 0$ . We also know the acceleration:

$$a_x = 0 \quad \text{and} \quad a_y = -9.80 \frac{\text{m}}{\text{s}^2} = -g$$

What else do we know? The gun is fired *horizontally* so that  $v_{0y} = 0$ , but we don't know  $v_{0x}$ . We don't know the time of flight but we do know that when  $x$  has the value 30 m then  $y$  has the value  $-1.9 \times 10^{-2}$  m. (Minus!)



Our equation for the  $y$  coordinate is

$$\begin{aligned} y &= y_0 + y_{0y}t + \frac{1}{2}a_y t^2 \\ &= 0 + 0 + \frac{1}{2}(-g)t^2 \\ &= -\frac{1}{2}gt^2 \end{aligned}$$

We can now ask: “At what time  $t$  does  $y$  equal  $-1.9 \times 10^{-2}$  m?” . Substitute  $y = -1.9 \times 10^{-2}$  m and solve:

$$t^2 = -\frac{2y}{g} = -\frac{2(-1.9 \times 10^{-2} \text{ m})}{9.80 \frac{\text{m}}{\text{s}^2}} = 3.9 \times 10^{-3} \text{ s}^2$$

which gives:

$$t = 6.2 \times 10^{-2} \text{ s}$$

Since this is the time of impact with the target, the time of flight of the bullet is  $t = 6.2 \times 10^{-2}$  s.

(b) The equation for  $x$ -motion is

$$\begin{aligned} x &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ &= 0 + v_{0x}t + 0 \\ &= v_{0x}t \end{aligned}$$

From part (a) we know that when  $t = 6.2 \times 10^{-2}$  s then  $x = 30$  m. This allows us to solve for  $v_{0x}$ :

$$v_{0x} = \frac{x}{t} = \frac{30 \text{ m}}{6.2 \times 10^{-2} \text{ s}} = 480 \frac{\text{m}}{\text{s}}$$

The muzzle velocity of the bullet is  $480 \frac{\text{m}}{\text{s}}$ .

6. In a local bar, a customer slides an empty beer mug on the counter for a refill. The bartender does not see the mug, which slides off the counter and strikes the floor 1.40 m from the base of the counter. If the height of the counter is 0.860 m, (a) with what speed did the mug leave the counter and (b) what was the direction of the mug’s velocity just before it hit the floor?

(a) The motion of the beer mug is shown in Fig. 3.2(a). We choose the origin of our  $xy$  coordinate system as being at the point where the mug leaves the counter. So the mug’s initial coordinates for its flight are  $x_0 = 0$ ,  $y_0 = 0$ .

At the very beginning of its motion through the air, the velocity of the mug is *horizontal*. (This is because its velocity was horizontal all the time it was sliding on the counter.) So we know that  $v_{0y} = 0$  but we don’t know the value of  $v_{0x}$ . (In fact, that’s what we’re trying to figure out!)

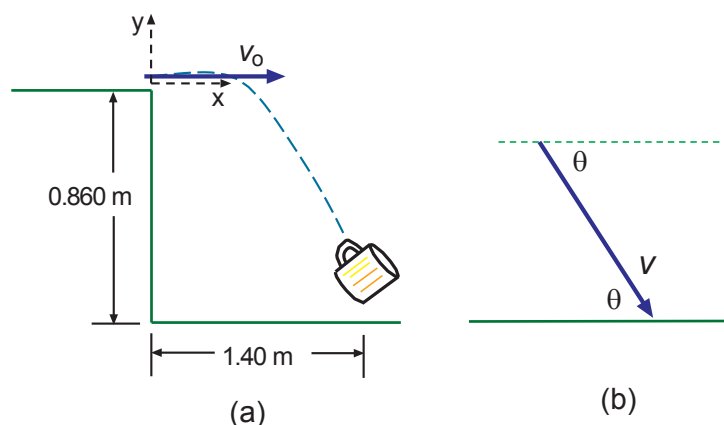


Figure 3.2: (a) Beer mug slides off counter and strikes floor! (b) Velocity vector of the beer mug at the time of impact.

We might begin by finding the time  $t$  at which the mug hit the floor. This is the time  $t$  at which  $y = -0.860$  m (recall how we chose the coordinates!), and we will need the  $y$  equation of motion for this; since  $v_{0y} = 0$  and  $a_y = -g$ , we get:

$$y = v_{0y}t + \frac{1}{2}a_y t^2 = -\frac{1}{2}gt^2$$

So we solve

$$-0.860\text{m} = -\frac{1}{2}gt^2$$

which gives

$$t^2 = \frac{2(0.860\text{m})}{g} = \frac{2(0.860\text{m})}{(9.80\frac{\text{m}}{\text{s}^2})} = 0.176\text{s}^2$$

so then

$$t = 0.419\text{s}$$

is the time of impact.

To find  $v_{0x}$  we consider the  $x$  equation of motion;  $x_0 = 0$  and  $a_x = 0$ , so we have

$$x = v_{0x}t.$$

At  $t = 0.419$  s we know that the  $x$  coordinate was equal to 1.40m. So

$$1.40\text{m} = v_{0x}(0.419\text{s})$$

Solve for  $v_{0x}$ :

$$v_{0x} = \frac{1.40\text{m}}{0.419\text{s}} = 3.34\frac{\text{m}}{\text{s}}$$

which tells us that the initial speed of the mug was  $v_0 = 3.34\frac{\text{m}}{\text{s}}$ .

(b) We want to find the components of the mug's velocity at the time of impact, that is, at  $t = 0.419$  s. Substitute into our expressions for  $v_x$  and  $v_y$ :

$$v_x = v_{0x} + a_x t = v_{0x} = 3.34\frac{\text{m}}{\text{s}}$$

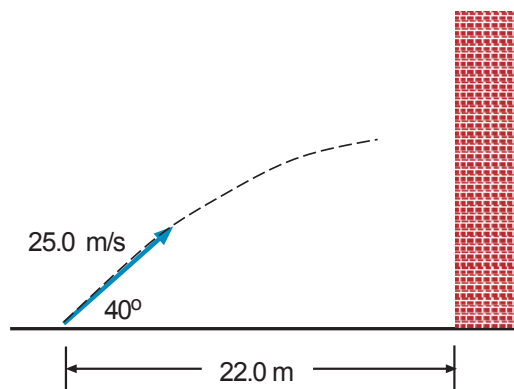


Figure 3.3: Ball is thrown toward wall at  $40^\circ$  above horizontal, in Example 7.

and

$$v_y = v_{0y} + a_y t = 0 + (-9.80 \frac{\text{m}}{\text{s}^2})(0.419 \text{ s}) = -4.11 \frac{\text{m}}{\text{s}} .$$

So at the time of impact, the *speed* of the mug was

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(3.34 \frac{\text{m}}{\text{s}})^2 + (-4.11 \frac{\text{m}}{\text{s}})^2} = 5.29 \frac{\text{m}}{\text{s}}$$

and, if as in Fig. 3.2(b) we let  $\theta$  be the angle *below the horizontal* at which the velocity vector is pointing, we see that

$$\tan \theta = \frac{4.11}{3.34} = 1.23 \quad \Rightarrow \quad \theta = \tan^{-1}(1.23) = 50.9^\circ .$$

At the time of impact, the velocity of the mug was directed at  $50.9^\circ$  below the horizontal.

7. You throw a ball with a speed of  $25.0 \frac{\text{m}}{\text{s}}$  at an angle of  $40.0^\circ$  above the horizontal directly toward a wall, as shown in Fig. 3.3. The wall is  $22.0 \text{ m}$  from the release point of the ball. (a) How long does the ball take to reach the wall? (b) How far above the release point does the ball hit the wall? (c) What are the horizontal and vertical components of its velocity as it hits the wall? (d) When it hits, has it passed the highest point on its trajectory?

(a) We will use a coordinate system which has its origin at the point of firing, which we take to be at ground level.

What is the mathematical condition which determines when the ball hits the wall? It is when the  $x$  coordinate of the ball is equal to  $22.0 \text{ m}$ . Then let's write out the  $x$ -equation of motion for the ball. The ball's initial  $x$ -velocity is

$$v_{0x} = v_0 \cos \theta_0 = (25.0 \frac{\text{m}}{\text{s}}) \cos 40.0^\circ = 19.2 \frac{\text{m}}{\text{s}}$$

and of course  $a_x = 0$ , so that the  $x$  motion is given by

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 = 19.2 \frac{\text{m}}{\text{s}} t$$

We solve for the time at which  $x = 22.0\text{m}$ :

$$x = 19.2 \frac{\text{m}}{\text{s}} t = 22.0\text{m} \quad \Rightarrow \quad t = \frac{22.0\text{m}}{19.2 \frac{\text{m}}{\text{s}}} = 1.15\text{s}$$

The ball hits the wall 1.15 s after being thrown.

(b) We will be able to answer this question if we can find the  $y$  *coordinate* of the ball at the time that it hits the wall, namely at  $t = 1.15\text{s}$ .

We need the  $y$  equation of motion. The initial  $y$  velocity of the ball is

$$v_{0y} = v_0 \sin \theta_0 = 25.0 \frac{\text{m}}{\text{s}} \sin 40.0^\circ = 16.1 \frac{\text{m}}{\text{s}}$$

and the  $y$  acceleration of the ball is  $a_y = -g$  giving:

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = 16.1 \frac{\text{m}}{\text{s}} t - \frac{1}{2}gt^2$$

which we use to find the  $y$  coordinate at  $t = 1.15\text{s}$ :

$$y = (16.1 \frac{\text{m}}{\text{s}})(1.15\text{s}) - \frac{1}{2}(9.80 \frac{\text{m}}{\text{s}^2})(1.15\text{s})^2 = 12.0\text{m}$$

which tells us that the ball hits the wall at 12.0m above the ground level (above the release point).

(c) The  $x$  and  $y$  components of the balls's velocity at the time of impact, namely at  $t = 1.15\text{s}$  are found from Eqs. 3.16:

$$v_x = v_{0x} + a_x t = 19.2 \frac{\text{m}}{\text{s}} + 0 = 19.2 \frac{\text{m}}{\text{s}}$$

and

$$v_y = v_{0y} + a_y t = 16.1 \frac{\text{m}}{\text{s}} + (-9.80 \frac{\text{m}}{\text{s}^2})(1.15\text{s}) = +4.83 \frac{\text{m}}{\text{s}} .$$

(d) Has the ball already passed the highest point on its trajectory? Suppose the ball was on its way *downward* when it struck the wall. Then the  $y$  component of the velocity would be *negative*, since it is always decreasing and at the trajectory's highest point it is zero. (Of course, the  $x$  component of the velocity stays the same while the ball is in flight.) Here we see that the  $y$  component of the ball's velocity is still *positive* at the time of impact. So the ball was still climbing when it hit the wall; it had *not* reached the highest point of its (free) trajectory.

8. The launching speed of a certain projectile is five times the speed it has at its maximum height. Calculate the elevation angle at launching.

We make a diagram of the projectile's motion in Fig. 3.4. The launch it speed is  $v_0$ , and the projectile is launched at an angle  $\theta_0$  upward from the horizontal.

We might start this problem by solving for the time it takes the projectile to get to maximum height, but we can note that at maximum height, there is no  $y$  velocity component, and

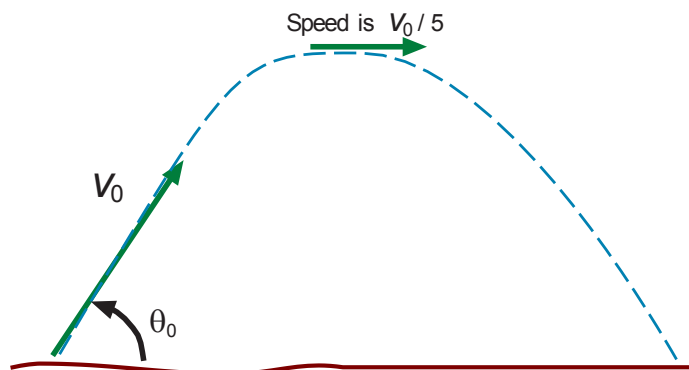


Figure 3.4: Motion of projectile in Example 8.

the  $x$  velocity component is *the same as it was when the projectile was launched*. Therefore at maximum height the velocity components are

$$v_x = v_0 \cos \theta_0 \quad \text{and} \quad v_y = 0$$

and so the speed of the projectile at maximum height is  $v_0 \cos \theta_0$ .

Now, we are told that the launching speed ( $v_0$ ) is five times the speed at maximum height. This gives us:

$$v_0 = 5v_0 \cos \theta_0 \quad \Rightarrow \quad \cos \theta_0 = \frac{1}{5}$$

which has the solution

$$\theta_0 = 78.5^\circ$$

So the elevation angle at launching is  $\theta_0 = 78.5^\circ$ .

9. A projectile is launched from ground level with speed  $v_0$  at an angle of  $\theta_0$  above the horizontal. Find: (a) the maximum height  $H$  attained by the projectile, and (b) the distance from the starting point at which the projectile strikes the ground; this is called the range  $R$  of the projectile.

**Comment:** This problem is worked in virtually every physics text, and it is sometimes simply called “The Projectile Problem”. I include it in this book for the sake of completeness and so that we can use the results if we need them later on. I do *not* treat it as part of the fundamental material of this chapter because it is a very particular application of free-fall motion. In this problem, the projectile impacts at the *same height* as the one from which it started, and that is *not* always the case. We must think about all projectile problems *individually* and not rely on simple formulae to plug numbers into!

The path of the projectile is shown in Fig. 3.5. The initial coordinates of the projectile are

$$x_0 = 0 \quad \text{and} \quad y_0 = 0,$$

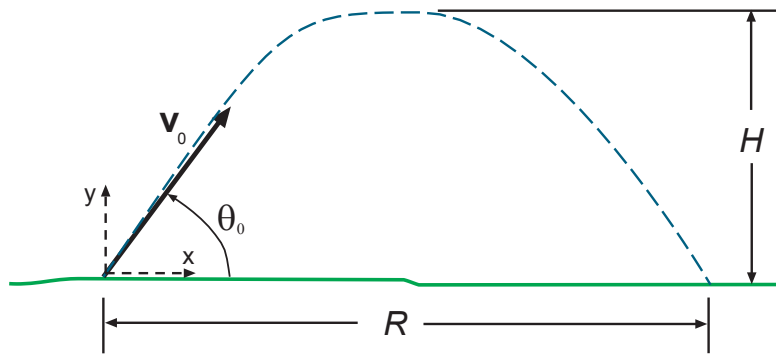


Figure 3.5: The common projectile problem; projectile is shot from ground level with speed  $v_0$  and angle  $\theta_0$  above the horizontal.

the components of the initial velocity are

$$v_{0x} = v_0 \cos \theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin \theta_0$$

and of course the (constant) acceleration of the projectile is

$$a_x = 0 \quad \text{and} \quad a_y = -g = -9.80 \frac{\text{m}}{\text{s}^2}$$

Then our equations for  $x(t)$ ,  $v_x(t)$ ,  $y(t)$  and  $v_y(t)$  are

$$v_x = v_0 \cos \theta_0$$

$$x = v_0 \cos \theta_0 t$$

$$v_y = v_0 \sin \theta_0 - gt$$

$$y = v_0 \sin \theta_0 t - \frac{1}{2}gt^2$$

(a) What does it mean for the projectile to get to “maximum height”? This is when it is neither increasing in height (rising) nor decreasing in height (falling); the vertical component of the velocity at this point is **zero**. At this particular time then,

$$v_y = v_0 \sin \theta_0 - gt = 0$$

so solving this equation for  $t$ , the projectile reaches maximum height at

$$t = \frac{v_0 \sin \theta_0}{g}$$

How high is the projectile at this time? To answer this, substitute this value of  $t$  into the equation for  $y$  and get:

$$\begin{aligned} y &= v_0 \sin \theta_0 \frac{v_0 \sin \theta_0}{g} - \frac{1}{2}g \left( \frac{v_0 \sin \theta_0}{g} \right)^2 \\ &= \frac{v_0^2 \sin^2 \theta_0}{g} - \frac{v_0^2 \sin^2 \theta_0}{2g} \\ &= \frac{v_0^2 \sin^2 \theta_0}{2g} \end{aligned}$$

This is the maximum height attained by the projectile:

$$H = \frac{v_0^2 \sin^2 \theta_0}{2g}$$

(b) What is the *mathematical* condition for when the projectile strikes the ground (since that is how we will find the range  $R$ )? We know that at this point, the projectile's  $y$  coordinate is zero:

$$y = v_0 \sin \theta_0 t - \frac{1}{2}gt^2 = 0$$

We want to solve this equation for  $t$ ; we can factor out  $t$  in this expression to get:

$$t(v_0 \sin \theta_0 - \frac{1}{2}gt) = 0$$

which has two solutions:

$$t = 0 \quad \text{and} \quad t = \frac{2v_0 \sin \theta_0}{g}$$

The first of these is just the time when the projectile was fired; yes,  $y$  *was* equal to zero then, but that's not what we want! The time at which the projectile strikes the ground is

$$t = \frac{2v_0 \sin \theta_0}{g}.$$

We want to find the value of  $x$  at the time of impact. Substituting this value of  $t$  into our equation for  $x(t)$ , we find:

$$\begin{aligned} x &= v_0 \cos \theta_0 \frac{2v_0 \sin \theta_0}{g} \\ &= \frac{2v_0^2 \sin \theta_0 \cos \theta_0}{g} \end{aligned}$$

This value of  $x$  is the range  $R$  of the projectile.

We can make this result a little simpler by recalling the trig relation:

$$\sin 2\theta_0 = 2 \sin \theta_0 \cos \theta_0.$$

Using this in our result for the range gives:

$$R = \frac{2v_0^2 \sin \theta_0 \cos \theta_0}{g} = \frac{v_0^2 \sin 2\theta_0}{g}$$

10. A projectile is fired in such a way that its horizontal range is equal to three times its maximum height. What is the angle of projection?

Now, this problem *does* deal with a projectile which starts and ends its flight at the same height, just as we calculated in the previous example. So we *can* use the results for the range  $R$  and maximum height  $H$  that we found there.

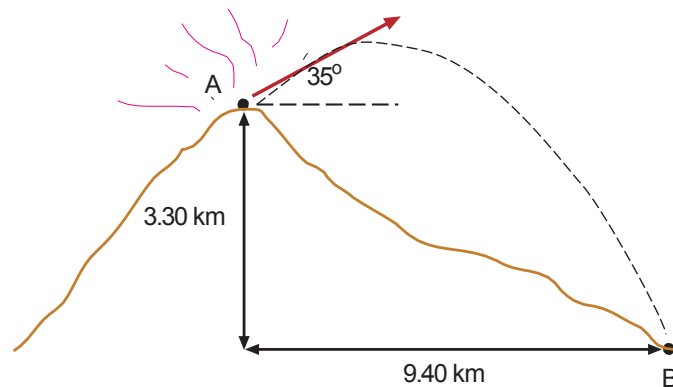


Figure 3.6: Volcanic bombs away!

The problem tells us that  $R = 3H$ . Substituting the expressions for  $H$  and  $R$  that we found in the last example (we pick the first expression we got for  $R$ ), we get:

$$R = \frac{2v_0^2 \sin \theta_0 \cos \theta_0}{g} = 3H = 3 \frac{v_0^2 \sin^2 \theta_0}{2g}$$

Cancelling stuff, we get:

$$2 \cos \theta_0 = \frac{3}{2} \sin \theta_0 \quad \Rightarrow \quad \tan \theta_0 = \frac{4}{3}$$

The solution is:

$$\theta_0 = \tan^{-1} (4/3) = 53.1^\circ$$

The projectile was fired at  $53.1^\circ$  above the horizontal.

11. During volcanic eruptions, chunks of solid rock can be blasted out of a volcano; these projectiles are called *volcanic bombs*. Fig. 3.6 shows a cross section of Mt. Fuji in Japan. (a) At what initial speed would the bomb have to be ejected, at  $35^\circ$  to the horizontal, from the vent at  $A$  in order to fall at the foot of the volcano at  $B$ ? (Ignore the effects of air on the bomb's travel.) (b) What would be the time of flight?

(a) We use a coordinate system with its origin at point  $A$  (the volcano "vent"); then for the flight from the vent at  $A$  to point  $B$ , the initial coordinates are  $x_0 = 0$  and  $y_0 = 0$ , and the final coordinates are  $x = 9.40 \text{ km}$  and  $y = -3.30 \text{ km}$ . Aside from this, we don't know the initial speed of the rock (that's what we're trying to find) or the time of flight from  $A$  to  $B$ . Of course, the acceleration of the rock is given by  $a_x = 0$ ,  $a_y = -g$ .

We start with the  $x$  equation of motion. The initial  $x$ -velocity is

$$v_{0x} = v_0 \cos \theta$$



where  $\theta = 35^\circ$  so the function  $x(t)$  is

$$\begin{aligned} x &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ &= 0 + v_0 \cos \theta t + 0 \\ &= v_0 \cos \theta t \end{aligned}$$

Now, we *do* know that at the time of impact  $x$  had the value  $x = 9.40\text{km}$  so if we now let  $t$  be the time of flight, then

$$(9.40\text{km}) = v_0 \cos \theta t \quad \text{or} \quad t = \frac{(9.40\text{km})}{v_0 \cos \theta} \quad (3.25)$$

Next we look at the  $y$  equation of motion. Since  $v_{0y} = v_0 \sin \theta$  we get:

$$\begin{aligned} y &= y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ &= 0 + v_0 \sin \theta t - \frac{1}{2}gt^2 \\ &= v_0 \sin \theta t - \frac{1}{2}gt^2 \end{aligned}$$

But at the time  $t$  of impact the  $y$  coordinate had the value  $y = -3.30\text{km}$ . If we also substitute for  $t$  in this expression using Eq. 3.25 we get:

$$\begin{aligned} -3.30\text{km} &= v_0 \sin \theta \frac{9.40\text{km}}{v_0 \cos \theta} - \frac{1}{2}g \left( \frac{9.40\text{km}}{v_0 \cos \theta} \right)^2 \\ &= (9.40\text{km}) \tan \theta - \frac{g(9.40\text{km})^2}{2v_0^2 \cos^2 \theta} \end{aligned}$$

At this point we are done with the *physics* problem. The *only* unknown in this equation is  $v_0$ , which we can find by doing a little algebra:

$$\begin{aligned} \frac{g(9.40\text{km})^2}{2v_0^2 \cos^2 \theta} &= (9.40\text{km}) \tan \theta + 3.30\text{km} \\ &= 9.88\text{km} \end{aligned}$$

which gives:

$$\begin{aligned} v_0^2 &= \frac{g(9.40 \text{ km})}{\cos^2 \theta (9.88 \text{ km})} \\ &= \frac{g(0.951 \text{ km})}{\cos^2 \theta} \\ &= \frac{(9.80 \frac{\text{m}}{\text{s}^2})(951 \text{ m})}{\cos^2 35^\circ} \\ &= 1.39 \times 10^4 \frac{\text{m}^2}{\text{s}^2} \end{aligned}$$

and finally

$$v_0 = 118 \frac{\text{m}}{\text{s}}$$

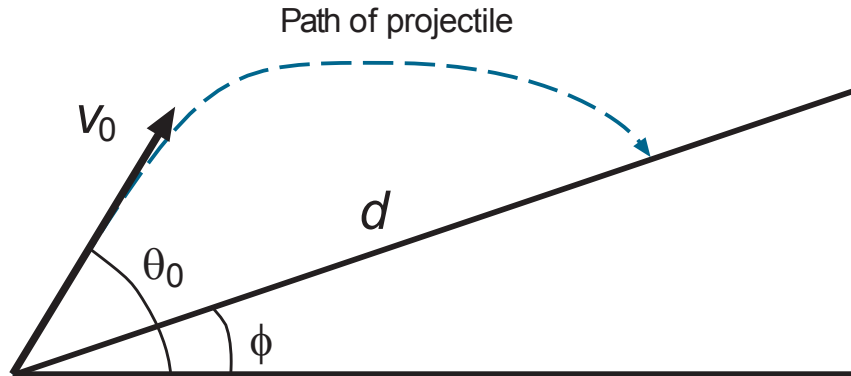


Figure 3.7: Projectile is fired up an incline, as described in Example 12

(b) Having  $v_0$  in hand, finding  $t$  is easy. Using our result from part(a) and Eq. 3.25 we find:

$$t = \frac{(9.40\text{km})}{v_0 \cos \theta} = \frac{(9400\text{m})}{(118 \frac{\text{m}}{\text{s}}) \cos 35^\circ} = 97.2\text{s}$$

The time of flight is 97.2s.

12. A projectile is fired up an incline (incline angle  $\phi$ ) with an initial speed  $v_0$  at an angle  $\theta_0$  with respect to the horizontal ( $\theta_0 > \phi$ ) as shown in Fig. 3.7. (a) Show that the projectile travels a distance  $d$  up the incline, where

$$d = \frac{2v_0^2 \cos \theta_0 \sin(\theta_0 - \phi)}{g \cos^2 \phi}$$

(b) For what value of  $\theta_0$  is  $d$  a maximum, and what is the maximum value?

(a) This is a relatively challenging problem, and of course it is completely analytic.

We can start by writing down equations for  $x$  and  $y$  as functions of time. By now we can easily see that we have:

$$\begin{aligned} x &= v_0 \cos \theta_0 t \\ y &= v_0 \sin \theta_0 t - \frac{1}{2}gt^2 \end{aligned}$$

We can combine these equations to get a relation between  $x$  and  $y$  for points on the trajectory; from the first, we have  $t = x/(v_0 \cos \theta_0)$ , and putting this into the second one gives:

$$\begin{aligned} y_{\theta_0} &= v_0 \sin \theta_0 \frac{x}{v_0 \cos \theta_0} - \frac{1}{2}g \left( \frac{x}{v_0 \cos \theta_0} \right)^2 \\ &= (\tan \theta_0)x - \frac{g}{2v_0^2 \cos^2 \theta_0} x^2 \end{aligned}$$

What is the condition for the time that the projectile hits the slope? Unlike the problems where a projectile impacts with the flat ground or a wall, we don't know the value of  $x$  or  $y$

at impact. But since the incline has a slope of  $\tan \varphi$ , the relation between  $x$  and  $y$  for points on the slope is

$$y = (\tan \varphi)x .$$

These two relations between  $x$  and  $y$  allow us to solve for the values of  $x$  and  $y$  where the impact occurs. Substituting for  $y$  above, we find:

$$(\tan \varphi)x = (\tan \theta_0)x - \frac{g}{2 v_0^2 \cos^2 \theta_0} x^2$$

A little rearranging gives:

$$\frac{g}{2 v_0^2 \cos^2 \theta_0} x + (\tan \varphi - \tan \theta_0)x = 0$$

and the solution for  $x$  is:

$$x = \frac{2v_0^2 \cos^2 \theta_0 (\tan \theta_0 - \tan \varphi)}{g}$$

The problem has us solve for the distance  $d$  up the slope; this distance is related to the impact value of  $x$  by:

$$d = \frac{x}{\cos \varphi}$$

and this gives us:

$$d = \frac{x}{\cos \varphi} = \frac{2v_0^2 \cos^2 \theta_0 (\tan \theta_0 - \tan \varphi)}{g \cos \varphi}$$

Although this is a perfectly good expression for  $d$ , it is not the one presented in the problem. (Among other things, it has another factor of  $\cos \varphi$  downstairs.) If we multiply top and bottom by  $\cos \varphi$  we find:

$$\begin{aligned} d &= \frac{2v_0^2 \cos^2 \theta_0 \cos \varphi (\tan \theta_0 - \tan \varphi)}{g \cos^2 \varphi} \\ &= \frac{2v_0^2 \cos \theta_0 (\cos \theta_0 \cos \varphi \tan \theta_0 - \cos \theta_0 \cos \varphi \tan \varphi)}{g \cos^2 \varphi} \\ &= \frac{2v_0^2 \cos \theta_0 (\cos \varphi \sin \theta_0 - \cos \theta_0 \sin \varphi)}{g \cos^2 \varphi} \end{aligned}$$

And now using an angle-addition identity from trigonometry in the numerator, we arrive at

$$d = \frac{2v_0^2 \cos \theta_0 \sin(\theta_0 - \varphi)}{g \cos^2 \varphi}$$

which is the preferred expression for  $d$ .

(b) In part (a) we found the up-slope impact distance as a function of launch angle  $\theta_0$ . (The launch speed  $v_0$  and the slope angle  $\varphi$  are taken to be fixed.) For a certain value of  $\theta_0$ ,

this function  $d(\theta_0)$  will take on a maximum value. To find this value, we differentiate the function  $d(\theta_0)$  and set the derivative equal to zero. We find:

$$\begin{aligned} d'(\theta_0) &= \frac{2v_0^2}{g \cos^2 \varphi} \frac{d}{d\theta_0} [\cos \theta_0 \sin(\theta_0 - \varphi)] \\ &= \frac{2v_0^2}{\cos^2 \varphi} [-\sin \theta_0 \sin(\theta_0 - \varphi) + \cos \theta_0 \cos(\theta_0 - \varphi)] g \\ &= \frac{2v_0^2}{g \cos^2 \varphi} \cos(2\theta_0 - \varphi) \end{aligned}$$

where in the last step we used the trig identity  $\cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos(\alpha + \beta)$ .

Now, to satisfy  $d'(\theta_0) = 0$  we must have  $\cos(2\theta_0 - \varphi) = 0$ . While this equation has infinitely many solutions for  $\theta_0$ , considering the values that  $\theta_0$  and  $\varphi$  may take on, we see that we need only look at the case where

$$2\theta_0 - \varphi = \frac{\pi}{2}$$

which of course, *does* solve the equation. This gives us:

$$\theta_0 = \frac{\pi}{4} + \frac{\varphi}{2}$$

for the value of  $\theta$  which makes the projectile go the farthest distance  $d$  up the slope.

To find what this value of  $d$  is, we substitute for  $\theta_0$  in our function  $d(\theta_0)$ . We find:

$$\begin{aligned} d_{\max} &= \frac{2v_0^2}{g \cos^2 \varphi} \cos \left( \frac{\pi}{4} + \frac{\varphi}{2} \right) \sin \left( \frac{\pi}{4} + \frac{\varphi}{2} - \varphi \right) \\ &= \frac{2v_0^2}{g \cos^2 \varphi} \cos \left( \frac{\pi}{4} + \frac{\varphi}{2} \right) \sin \left( \frac{\pi}{4} - \frac{\varphi}{2} \right) \end{aligned}$$

This expression is *correct* but it can be simplified. We use the trig identity which states:

$$\sin \alpha \cos \beta = \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta)$$

this gives us:

$$\begin{aligned} \sin \left( \frac{\pi}{4} - \frac{\varphi}{2} \right) \cos \left( \frac{\pi}{4} + \frac{\varphi}{2} \right) &= \frac{1}{2} \sin \frac{\pi}{4} + \frac{1}{2} \sin(-\varphi) \\ &= \frac{1}{2} - \frac{1}{2} \sin \varphi \\ &= \frac{1}{2} (1 - \sin \varphi) \end{aligned}$$

which *is* a lot simpler. Using this result in our expression for  $d_{\max}$  gives:

$$d_{\max} = \frac{2v_0^2}{g \cos^2 \varphi} \frac{(1 - \sin \varphi)}{2} = \frac{v_0^2 (1 - \sin \varphi)}{g(1 - \sin^2 \varphi)} = \frac{v_0^2}{g(1 + \sin \varphi)}$$

which is as simple as it's going to get!

We can *check* result for a couple well-known cases. If  $\varphi = 0$  we are dealing with the common projectile problem on level ground for which we know we get maximum range when  $\theta_0 = 45^\circ$  and from our solution for that problem we get  $R = \frac{v_0^2}{g}$ . If  $\varphi = 90^\circ$  we have the problem of a projectile fired straight up; one can show that the maximum height reached is  $H = \frac{v_0^2}{2g}$  which again agrees with the formula we've derived.

## 3.2.5 Uniform Circular Motion

13. In one model of the hydrogen atom, an electron orbits a proton in a circle of radius  $5.28 \times 10^{-11}$  m with a speed of  $2.18 \times 10^6 \frac{\text{m}}{\text{s}}$ . (a) What is the acceleration of the electron in this model? (b) What is the period of the motion?

(a) The electron moves in a circle with constant speed. It is accelerating *toward the center of the circle* and the acceleration has magnitude  $a_{\text{cent}} = \frac{v^2}{r}$ . Substituting the given values, we have:

$$a_{\text{cent}} = \frac{v^2}{r} = \frac{(2.18 \times 10^6 \frac{\text{m}}{\text{s}})^2}{(5.28 \times 10^{-11} \text{ m})} = 9.00 \times 10^{22} \frac{\text{m}}{\text{s}^2}$$

The acceleration has magnitude  $9.00 \times 10^{22} \frac{\text{m}}{\text{s}^2}$ .

(b) As the electron makes one trip around the circle of radius  $r$ , it moves a distance  $2\pi r$  (the circumference of the circle). If  $T$  is the period of the motion, then the speed of the electron is given by the ratio of distance to time,

$$v = \frac{2\pi r}{T} \quad \text{which gives...} \quad T = \frac{2\pi r}{v}$$

which shows why Eq. 3.22 is true. Substituting the given values, we get:

$$T = \frac{2\pi(5.28 \times 10^{-11} \text{ m})}{(2.18 \times 10^6 \frac{\text{m}}{\text{s}})} = 1.52 \times 10^{-16} \text{ s}$$

The period of the electron's motion is  $1.52 \times 10^{-16}$  s.

14. A rotating fan completes 1200 revolutions every minute. Consider a point on the tip of a blade, at a radius of 0.15 m. (a) Through what distance does the point move in one revolution? (b) What is the speed of the point? (c) What is its acceleration? (d) What is the period of the motion?

(a) As the fan makes one revolution, the point in question moves through a circle of radius 0.15 m so the distance it travels is the circumference of that circle, i.e.

$$d = 2\pi r = 2\pi(0.15 \text{ m}) = 0.94 \text{ m}$$

The point travels 0.94 m.

(b) If in one minute (60 s) the fan makes 1200 revolutions, the time to make *one* revolution must be

$$\text{Time for one rev} = T = \frac{1}{1200} \cdot (1.00 \text{ min}) = \frac{1}{1200} \cdot (60.0 \text{ s}) = 5.00 \times 10^{-2} \text{ s}$$

Using our answer from part (a), we know that the point travels 0.94 m in  $5.000 \times 10^{-2}$  s, moving at constant speed. Therefore that speed is:

$$v = \frac{d}{T} = \frac{0.94 \text{ m}}{5.000 \times 10^{-2} \text{ s}} = 19 \frac{\text{m}}{\text{s}}$$

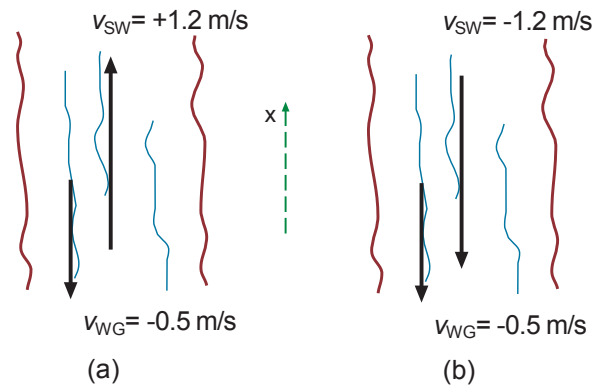


Figure 3.8: (a) Velocities for case where swimmer swims upstream. (b) Velocities for case where swimmer swims downstream.

(c) The point is undergoing uniform circular motion; its acceleration is always *toward the center* and has magnitude  $a_{\text{cent}} = \frac{v^2}{r}$ . Substituting,

$$a_{\text{cent}} = \frac{v^2}{r} = \frac{(19 \frac{\text{m}}{\text{s}})^2}{(0.15 \text{ m})} = 2.4 \times 10^3 \frac{\text{m}}{\text{s}^2}$$

(d) The period of the motion is the time for the fan to make one revolution. And we already found this in part (b)! It is:

$$T = 5.00 \times 10^{-2} \text{ s}$$

### 3.2.6 Relative Motion

15. A river has a steady speed of  $0.500 \frac{\text{m}}{\text{s}}$ . A student swims upstream a distance of  $1.00 \text{ km}$  and returns to the starting point. If the student can swim at a speed of  $1.20 \frac{\text{m}}{\text{s}}$  in still water, how long does the trip take? Compare this with the time the trip would take if the water were still.

What happens if the water *is* still? The student swims a distance of  $1.00 \text{ km}$  “upstream” at a speed of  $1.20 \frac{\text{m}}{\text{s}}$ ; using the simple distance/time formula  $d = vt$  the time for the trip is

$$t = \frac{d}{v} = \frac{1.0 \times 10^3 \text{ m}}{1.20 \frac{\text{m}}{\text{s}}} = 833 \text{ s}$$

and the same is true for the trip back “downstream”. So the total time for the trip is

$$833 \text{ s} + 833 \text{ s} = 1.67 \times 10^3 \text{ s} = 27.8 \text{ min}$$

Good enough, but what about the case where the water is *not* still? And what does that have to do with relative velocities? In Fig. 3.8, the river is shown; it flows in the  $-x$

direction. At all times, the velocity of the water *with respect to the ground* is

$$v_{WG} = -0.500 \frac{\text{m}}{\text{s}} .$$

When the student swims upstream, as represented in Fig. 3.8(a), his velocity *with respect to the water* is

$$v_{SW} = +1.20 \frac{\text{m}}{\text{s}} .$$

We know this because we are given his swimming speed for *still* water.

Now we are interested in the student's velocity *with respect to the ground*, which we will call  $v_{SG}$ . It is given by the sum of his velocity with respect to the water and the water's velocity with respect to the ground:

$$v_{SG} = v_{SW} + v_{WG} = +1.20 \frac{\text{m}}{\text{s}} - 0.500 \frac{\text{m}}{\text{s}} = 0.70 \frac{\text{m}}{\text{s}}$$

and so to cover a displacement of  $\Delta x = 1.00\text{km}$  (measured along the ground!) requires a time

$$\Delta t = \frac{\Delta x}{v_{SG}} = \frac{1.00 \times 10^3 \text{ m}}{0.70 \frac{\text{m}}{\text{s}}} = 1.43 \times 10^3 \text{ s}$$

Then the student swims downstream (Fig. 3.8(b)) and his velocity with respect to the water is

$$v_{SW} = -1.20 \frac{\text{m}}{\text{s}}$$

giving him a velocity with respect to the ground of

$$v_{SG} = v_{SW} + v_{WG} = -1.20 \frac{\text{m}}{\text{s}} - 0.500 \frac{\text{m}}{\text{s}} = -1.70 \frac{\text{m}}{\text{s}}$$

so that the time to cover a displacement of  $\Delta x = -1.00\text{km}$  is

$$\Delta t = \frac{\Delta x}{v_{SG}} = \frac{(-1.00 \times 10^3 \text{ m})}{(-1.70 \frac{\text{m}}{\text{s}})} = 5.88 \times 10^2 \text{ s}$$

The *total* time to swim upstream and then downstream is

$$\begin{aligned} t_{\text{Total}} &= t_{\text{up}} + t_{\text{down}} \\ &= 1.43 \times 10^3 \text{ s} + 5.88 \times 10^2 \text{ s} = 2.02 \times 10^3 \text{ s} = 33.6 \text{ min} . \end{aligned}$$

16. A light plane attains an airspeed of  $500\text{km/hr}$ . The pilot sets out for a destination  $800\text{km}$  to the north but discovers that the plane must be headed  $20.0^\circ$  east of north to fly there directly. The plane arrives in  $2.00\text{hr}$ . What was the wind velocity vector?

Whoa! What the Hell is this problem talking about???

When a plane flies in air which *itself* is moving (i.e. there is a wind velocity) there are three (vector) velocities we need to think about; I will refer to them as:

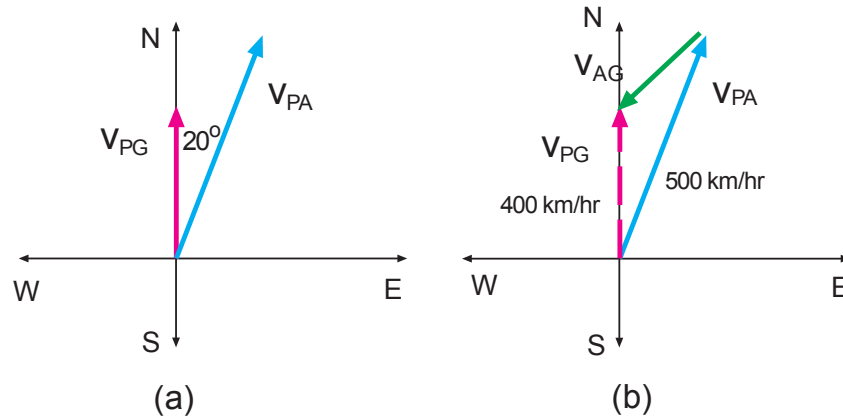


Figure 3.9: (a) Vectors for the plane's velocity with respect to the ground ( $v_{PG}$ ) and with respect to the moving air ( $v_{PA}$ ). (b) The sum of the plane's velocity relative to the air and the wind velocity gives the plane's velocity with respect to the ground,  $v_{PG}$ .

$v_{PA}$ : Velocity of the plane with respect to the air. The magnitude of this vector is the "airspeed" of the plane. (This is the only thing that a plane's "speedometer" can really measure.)

$v_{AG}$ : Velocity of the air with respect to the ground. This is the wind velocity.

$v_{PG}$ : Velocity of the plane with respect to the ground. This is the quantity which tells us the rate of (ground!) travel of the plane.

These three vectors are related via:

$$v_{PG} = v_{PA} + v_{AG}$$

The first thing we are given in this problem is that the magnitude of  $v_{PA}$  is 500 km/hr. The plane needs to fly *due north* and this tells us that  $v_{PG}$  (the real direction of motion of the plane) points north (along the  $y$  axis). We are then told that the plane's "heading" is  $20.0^\circ$  east of north, which tells us that the direction of  $v_{PA}$  lies in this direction. These facts are illustrated in Fig. 3.9(a).

Now if the plane travels 800km in 2.00hr then its speed (with respect to the ground!) is

$$v_{PG} = \frac{800\text{km}}{2.00\text{hr}} = 400 \frac{\text{km}}{\text{hr}} .$$

which we note in Fig. 3.9(b). Since we now have the magnitudes and directions of  $v_{PA}$  and  $v_{PG}$  we can compute the wind velocity,

$$v_{AG} = v_{PG} - v_{PA}$$

The  $x$  component of this vector is

$$v_{AG,x} = 0 - 500 \frac{\text{km}}{\text{hr}} \sin 20.0^\circ = -171 \frac{\text{km}}{\text{hr}}$$

and its  $y$  component is

$$v_{AG,y} = 400 - 500 \frac{\text{km}}{\text{hr}} \cos 20.0^\circ = -69.8 \frac{\text{km}}{\text{hr}}$$



So the wind velocity is

$$v_{AG} = -171 \frac{\text{km}}{\text{hr}} \mathbf{i} - 69.8 \frac{\text{km}}{\text{hr}} \mathbf{j}$$

If we want to express the velocity as a magnitude and direction, we find:

$$v_{AG} = \sqrt{(-171 \frac{\text{km}}{\text{hr}})^2 + (-69.8 \frac{\text{km}}{\text{hr}})^2} = 185 \frac{\text{km}}{\text{hr}}$$

so the wind speed is  $185 \frac{\text{km}}{\text{hr}}$ . The direction of the wind, measured as an angle  $\theta$  counter-clockwise from the east is found from its components:

$$\tan \theta = \frac{-69.8}{-171} = 0.408 \quad \Rightarrow \quad \theta = \tan^{-1}(0.408) = 22^\circ$$

(Here we have made sure to get the angle right! Since both components are negative,  $\theta$  lies in the third quadrant!) Since  $180^\circ$  would be *due West* and the wind direction is  $22^\circ$  larger than that, we can also say that the wind direction is “ $22^\circ$  south of west”.